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Master-Slave interaction of heteroclinic networks

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DEDICATORIA

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Abstract

Mathematical models of cognitive processes possess very often a heteroclinic networks that are images of sequential activity of the systems. One of the most important problems for such systems is to study an interaction of heteroclinic networks. The simplest coupling is of the master-slave type when one system evolves independently and another one is subjected to perturbations generated by the first one. It is supposed to consider such a problem in the thesis for the case when both systems have heteroclinic cycles consisting of saddle equilibrium points and joining them heteroclinic trajectory. Both systems are of the generalized Lotka-Volterra type, main results include conditions of existing a heteroclinic network resulting from interacted systems and, also, a discovery a two-dimensional heteroclinic attractor. The thesis will include both theoretical and numerical studies.

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Chapter 1

Introduction

Mathematical models of cognitive processes possess very often a heteroclinic networks that are images of sequential activity of the systems [2, 3, 8–11]. One of the most important problems for such systems is to study an interaction of heteroclinic networks. For it, we consider a system S of coupled subsystems. This system S is of master-slave type i.e one of the subsystems evolves independently and another one is subjected to perturbations generated by the first one. The goal of this work is to study the dynamics of system S in a specific situation when both subsystems are of the Lotka-Volterra type. We will impose conditions under which the system S has a heteroclinic network and, also, additional conditions such that this network serves as a skeleton of a two-dimensional heteroclinic attractor. These are main results of the work.

This work is organized as follows. In chapter 2 we give some important definitions which will be used in this thesis. Definition of heteroclinic network is given in this chapter. In chapter 3 we will study analytically the master-slave system, formulate some propositions, collolaries and finaly the theorem which tells that under the conditions given in it, in the full space the master-slave system has a heteroclinic network. In this chapter we can find the conditions in which each equilibrium point of the heteroclinic network has two-dimensional unstable manifold.

In chapter 4 we present numerical results. We choose values of σ_i, δ_i and η_{ij}, then we find values of ρ_{ij} and ξ_{ij} which satisfy the conditions for the existence of heteroclinic network. We substitute this parameters to master-slave system and compute it. The portrait of phase space is shown and one can see that, in fact we obtain the heteroclinic network. For numerical work first we start with $\eta_{ij} \equiv 0$, and then we take $\eta_{ij} \neq 0$(but very small). In chapter 5 we impose conditions under which the uncoupled system has a two-dimensional heteroclinic attractor homeomorphic to two-dimensional non-smooth torus and prove that this torus still exists if the values of coupling parameters are small enough.

Future work is suggested in chapter 6.

Chapter 2

Definitions

First, we remind known notions we should use bellow.

Definition 2.0.1 *The* ***Jacobian matrix*** *is the matrix of all first-order partial derivatives of a vector-valued function. Suppose $f : R^n \to R^m$ is a function which takes as input the vector $x \in R^n$ and produces as output the vector $f(x) \in R^m$. Then the Jacobian matrix J of f is a mxn matrix, usually defined and arranged as follows:*

$$J = \frac{df}{dx} = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{2.1}$$

Definition 2.0.2 *Let $A \in M_{nxn}(F)$. $0 \neq x \in F^n$ is said to be an* ***eigenvector*** *of A if $\exists \lambda \in F$ such that $Ax = \lambda x$. λ is called the corresponding* ***eigenvalue*** *the eigenvector x.*

Definition 2.0.3 *If $A \in M_{nxn}(F)$, the polynomial $det(A - tI_n)$ in the unknown t is called* ***characteristic polynomial*** *of A.*

The following statements are well known:

1) The characteristic polynomial of $A \in M_{nxn}(F)$ is a polynomial of degree n with leading coefficient $(-1)^n$.

2) Let $A \in M_{nxn}(F)$ and let $f(t)$ the characteristic polynomial of A. Then
a)A scalar λ is an eigenvalue of A if and only if $f(\lambda) = 0$.
b)The maximal number of different eigenvalues of A is n.

Let us consider a system of differential equations

$$\dot{x} = X(x) \tag{2.2}$$

where $x \in R^n$ and X is a smooth function in some region $D \subset R^n$.

A trajectory $\{x(t)\}$ of system (2.2) is called an equilibrium state if it does not depend on time, i.e $x(t) \equiv x_0 = const$. It follows from the definition that the coordinates of the equilibrium state can be found as the solution of the system

$$X(x_0) = 0. \tag{2.3}$$

The study of system (2.2) near an equilibrium state is based on a standard linearization producere (see [5]). Linearization of system (2.2) has the form

$$\dot{y} = Ay \tag{2.4}$$

where $A = \frac{\partial X(x_0)}{\partial x}$ and $y = x - x_0$.

The stability of an equilibrium state is determined by eigenvalues $(\lambda_1, ..., \lambda_n)$ of the Jacobian matrix A(see [5]). An equilibrium point is called the saddle if it has eigenvalues with both negative and positive real parts.

Definition 2.0.4 *A trajectory $\{x(t)\}$ is called the heteroclinic one if $\lim\limits_{t\to\infty} x(t) = x_0$, $\lim\limits_{t\to-\infty} x(t) = y_0$ where x_0, y_0 are different equilibrium points.*

Definition 2.0.5 *The set $\Gamma := \bigcup\limits_{k=1}^{n} O_k \bigcup\limits_{k=1}^{n-1} \Gamma_{k\,k+1}$ where O_k are saddle equilibrium points and $\Gamma_{k\,k+1}$ is a heteroclinic trajectory joining O_k and O_{k+1} is called **heteroclinic network**. If, in addition, O_n is connected to O_1 by a heteroclinic trajectory, then the heteroclinic network is called **heteroclinic cycle**.*

Let $Re\lambda_i < 0$, $i = 1, ..., m$, $Re\lambda_j > 0$, $j = m+1, ..., n$ for a saddle O.

Definition 2.0.6 *The saddle O is said to be dissipative if $\nu = -\frac{\max\limits_{1\le i\le m} Re\lambda_i}{\max\limits_{m+1\le j\le n} Re\lambda_j} > 1$. ν is called the saddle value of O.*

It is known (see, for instance [2, 3]) that if the saddle values of all saddles in a heteroclinic cycle greater than 1, then the heteroclinic cycle is an attractor, i.e. it attracts trajectories going through an open set of initial points.

Chapter 3

Heteroclinic network for a master-slave system

Let us start with a formulation of the following problem. Consider a system of coupled susbsystems

$$\dot{x} = F_1(x) \tag{3.1}$$

$$\dot{y} = F_2(y) + \epsilon \ G_2(x, y) \tag{3.2}$$

It is said to be of master-slave type, where $x-$ master coordinates, $y-$ slave coordinates. The problem is to study dynamics of system (3.1),(3.2) depending on values of coupling and other parameters. The goal of this work is to study the dynamics of system (3.1),(3.2) in a specific situation when the systems (3.1),(3.2) are of the Lotka-Volterra type. So, we consider the system

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{12}x_2 - \rho_{13}x_3) \\ \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{21}x_1 - \rho_{23}x_3) \\ \dot{x_3} = x_3(\sigma_3 - x_3 - \rho_{31}x_1 - \rho_{32}x_2) \end{cases} \tag{3.3}$$

$$\begin{cases} \dot{y_1} = y_1(\delta_1 - y_1 - \xi_{12}y_2 - \xi_{13}y_3 - \eta_{11}x_1 - \eta_{12}x_2 - \eta_{13}x_3) \\ \dot{y_2} = y_2(\delta_2 - y_2 - \xi_{21}y_1 - \xi_{23}y_3 - \eta_{21}x_1 - \eta_{22}x_2 - \eta_{23}x_3) \\ \dot{y_3} = y_3(\delta_3 - y_3 - \xi_{31}y_1 - \xi_{32}y_2 - \eta_{31}x_1 - \eta_{32}x_2 - \eta_{33}x_3) \end{cases} \tag{3.4}$$

where $0 < \sigma_1 < \sigma_2 < \sigma_3, 0 < \rho_{ij}, 0 < \delta_1 < \delta_2 < \delta_3, 0 < \xi_{ij}$ and $0 \le \eta_{ij}$.

3.1 Study of two-dimensional subsystems

This system (3.3),(3.4) has many invariant two-dimensional planes such that the coordinates except for two of them are iqual to 0. The corresponding plane is invariant, and the corresponding 2-dimensional subsystem has the form

$$\begin{cases} \dot{u_1} = u_1(\alpha_1 - u_1 - \beta_1 u_2) \\ \dot{u_2} = u_2(\alpha_2 - u_2 - \beta_2 u_1) \end{cases} \tag{3.5}$$

The system (3.5) has three equilibrium points: the origin $O = (0, 0), E_1 = (\alpha_1, 0)$ and $E_2 = (0, \alpha_2)$.

Proposition 3.1.1 *Under the conditions*

$$\alpha_2 - \beta_2\alpha_1 > 0, \tag{3.6}$$

$$\alpha_1 - \beta_1\alpha_2 < 0, \tag{3.7}$$

the point E_1 is a saddle equilibrium with eigenvalues $\lambda(E_1)_1 = -\alpha_1, \lambda(E_1)_2 = \alpha_2 - \beta_2\alpha_1$ and the point E_2 is a node equilibrium with eigenvalues $\lambda(E_2)_1 = -\alpha_2, \lambda(E_2)_2 = \alpha_1 - \beta_1\alpha_2$.

Proof. Jacobian matrix for system (3.5) is

$$J_{\dot{u}_i} = \begin{pmatrix} \alpha_1 - 2u_1 - \beta_1 u_2 & -\beta_1 u_1 \\ -\beta_2 u_2 & \alpha_2 - 2u_2 - \beta_2 u_1 \end{pmatrix} \tag{3.8}$$

For E_1 we have

$$J_{E_1} = \begin{pmatrix} -\alpha_1 & -\beta_1\alpha_1 \\ 0 & \alpha_2 - \beta_2\alpha_1 \end{pmatrix} \tag{3.9}$$

Thus eigenvalues are $\lambda(E_1)_1 = -\alpha_1, \lambda(E_1)_2 = \alpha_2 - \beta_2\alpha_1$. $\lambda(E_1)_1 < 0$ because $\alpha_1 > 0$ and $\lambda(E_1)_2 > 0$ because of (3.6). It implies that E_1 is a saddle equilibrium point.

Now for E_2 we have

$$J_{E_2} = \begin{pmatrix} \alpha_1 - \beta_1\alpha_2 & 0 \\ -\beta_2\alpha_2 & -\alpha_2 \end{pmatrix} \tag{3.10}$$

Thus eigenvalues are $\lambda(E_2)_1 = -\alpha_2, \lambda(E_2)_2 = \alpha_1 - \beta_1\alpha_2$. $\lambda(E_2)_1 < 0$ because $\alpha_2 > 0$ and $\lambda(E_2)_2 < 0$ because of (3.7). It implies that E_2 is a node equilibrium point. ∎

Proposition 3.1.2 *Consider the system (3.5). Assume that*
i) $1 - \beta_1\beta_2 \neq 0$.
ii) conditions (3.6),(3.7) are satisfaced.
Then there exists a heteroclinic trajectory $\Gamma_{E_1E_2}$ joining $E_1 = (\alpha_1, 0)$ and $E_2 = (0, \alpha_2)$.

Proof. Let D be the region $D = \{0 < u_1 \leq \alpha_1, 0 < u_2 \leq \alpha_2\}$. It is simple to see that D is an absorbing region. Assume that there exists an equilibrium point $E^\star = (u_{1_0}, u_{2_0}) \in D$, and that $i)$ is satisfaced. Then $E^\star = (\frac{\alpha_1 - \beta_1\alpha_2}{1-\beta_1\beta_2}, \frac{\alpha_2 - \beta_2\alpha_1}{1-\beta_1\beta_2})$. Now $ii)$ implies that $u_{1_0} < 0$ or $u_{2_0} < 0$ which is a contradition because $E^\star \in D$. Therefore there are not equilibrium points inside of D except for E_1 and E_2 which are saddle and stable node respectively because of Proposition 3.1.1. The set of limit points of the unstable trajectory with the start point E_1 could be either periodic trajectory or must contain an equilibrium point(see [12]). But if it is a periodic trajectory then, inside it, there exists an equilibrium point which is impossible, as was just shown. The only admissible equilibrium point is E_2. The sentence is true for the second logical possibility. Thus there exists a heteroclinic trajectory $\Gamma_{E_1E_2}$ inside of D joining E_1 and E_2. ■

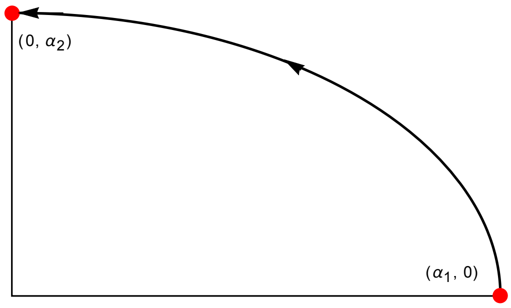

Figure 3.1: Heteroclinic trajectory joining E_1 and E_2.

3.2 Study of three-dimensional subsystems

We apply proposition 3.1.1 to study, first, the system (3.3). We are interested in equilibrium points $O_1 = (\sigma_1, 0, 0), O_2 = (0, \sigma_2, 0)$ and $O_3 = (0, 0, \sigma_3)$. The direct corollary of Propositions 3.1.1,3.1.2 will be

Corollary 3.2.1 *Under the conditions*

$$\sigma_2 - \rho_{21}\sigma_1 > 0, \quad \sigma_3 - \rho_{31}\sigma_1 < 0, \tag{3.11}$$

$$\sigma_3 - \rho_{32}\sigma_2 > 0, \quad \sigma_1 - \rho_{12}\sigma_2 < 0, \tag{3.12}$$

$$\sigma_1 - \rho_{13}\sigma_3 > 0, \quad \sigma_2 - \rho_{23}\sigma_3 < 0, \tag{3.13}$$

$$1 - \rho_{12}\rho_{21} \neq 0, \tag{3.14}$$

$$1 - \rho_{23}\rho_{32} \neq 0, \tag{3.15}$$

$$1 - \rho_{31}\rho_{13} \neq 0, \tag{3.16}$$

the master system (3.3) has a heteroclinic cycle coinsisting of equilibrium points O_1, O_2, O_3 *and heteroclinic trajectories* $\Gamma_{12}, \Gamma_{23}, \Gamma_{31}$.

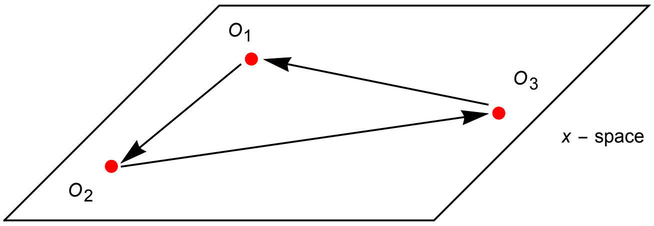

Figure 3.2: Heteroclinic cycle of master system.

Proof. First we have that there are equilibrium points for master system (3.3) which are $O = (0, 0, 0)$, $O_1 = (\sigma_1, 0, 0), O_2 = (0, \sigma_2, 0)$ and $O_3 = (0, 0, \sigma_3)$. Now the jacobian matrix of master system is

$$J_{\dot{x}_i} = \begin{pmatrix} \sigma_1 - 2x_1 - \rho_{12}x_2 - \rho_{13}x_3 & -\rho_{12}x_1 & -\rho_{13}x_3 \\ -\rho_{21}x_2 & \sigma_2 - 2x_2 - \rho_{21}x_1 - \rho_{23}x_3 & -\rho_{23}x_2 \\ -\rho_{31}x_3 & -\rho_{32}x_3 & \sigma_3 - 2x_3 - \rho_{31}x_1 - \rho_{32}x_2 \end{pmatrix} \tag{3.17}$$

For O_1

$$J_{O_1} = \begin{pmatrix} -\sigma_1 & -\rho_{12}\sigma_1 & 0 \\ 0 & \sigma_2 - \rho_{21}\sigma_1 & 0 \\ 0 & 0 & \sigma_3 - \rho_{31}\sigma_1 \end{pmatrix} \tag{3.18}$$

Thus the eigenvalues are $\lambda_1{}^{(1)}$=-σ_1 ,$\lambda_2{}^{(1)}$= σ_2-ρ_{21} σ_1 and $\lambda_3{}^{(1)}$= σ_3-ρ_{31} σ_1. Now because of (3.11) we have that O_1 is a saddle equlilibrium point with one-dimensional unstable manifold.

For O_2

$$J_{O_2} = \begin{pmatrix} \sigma_1 - \rho_{12}\sigma_2 & 0 & 0 \\ -\rho_{21}\sigma_2 & -\sigma_2 & -\rho_{23}\sigma_2 \\ 0 & 0 & \sigma_3 - \rho_{32}\sigma_2 \end{pmatrix} \tag{3.19}$$

Thus the eigenvalues are $\lambda_1{}^{(2)}$=-σ_2 ,$\lambda_2{}^{(2)}$= σ_3-ρ_{32} σ_2 and $\lambda_3{}^{(2)}$= σ_1-ρ_{12} σ_2. Because of (3.12) we have that O_2 is a saddle equlilibrium point with one-dimensional unstable manifold.

Finally for O_3

$$J_{O_3} = \begin{pmatrix} \sigma_1 - \rho_{13}\sigma_3 & 0 & 0 \\ 0 & \sigma_2 - \rho_{23}\sigma_3 & 0 \\ -\rho_{31}\sigma_3 & -\rho_{32}\sigma_3 & -\sigma_3 \end{pmatrix} \tag{3.20}$$

Thus the eigenvalues are $\lambda_1{}^{(3)}$=-σ_3 ,$\lambda_2{}^{(3)}$= σ_1-ρ_{13} σ_3 and $\lambda_3{}^{(3)}$= σ_2-ρ_{23} σ_3. Because of (3.13) we have that O_3 is a saddle equlilibrium point with one-dimensional unstable manifold.

Let $P_{x_ix_j}$ be the x_ix_j plane. Consider $P_{x_1x_2}$, thus the restriction of master system (3.3) on this plane is the following

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{12}x_2) \\ \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{21}x_1) \end{cases} \tag{3.21}$$

Because of Proposition 3.1.2, under the conditions (3.11),(3.12) and (3.14) there exists a heteroclinic trayectory Γ_{12} joining O_1 and O_2.

Consider $P_{x_2x_3}$, thus the restriction of master system (3.3) on this plane is the following

$$\begin{cases} \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{23}x_3) \\ \dot{x_3} = x_3(\sigma_3 - x_3 - \rho_{32}x_2) \end{cases} \tag{3.22}$$

Because of Proposition 3.1.2, under the conditions (3.12),(3.13) and (3.15) there exists a heteroclinic trayectory Γ_{23} joining O_2 and O_3.

Finally consider the system

$$\begin{cases} \dot{x_3} = x_3(\sigma_3 - x_3 - \rho_{31}x_1) \\ \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{13}x_3) \end{cases} \tag{3.23}$$

Under the conditions (3.11),(3.13) and (3.16) there exists a heteroclinic trayectory Γ_{31} joining O_3 and O_1 because of Proposition 3.1.2. So the set $\Gamma_3 = (O_1 \cup O_2 \cup O_3) \cup (\Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31})$ is a heteroclinic cycle for master system. ■

Now, we consider the slave system (3.4) provided that coupling coefficients are equal to 0.

Proposition 3.2.1 *Assume that $\eta_{ij} \equiv 0$. Under the conditions*

$$\delta_2 - \xi_{21}\delta_1 > 0, \quad \delta_3 - \xi_{31}\delta_1 < 0, \tag{3.24}$$

$$\delta_3 - \xi_{32}\delta_2 > 0, \quad \delta_1 - \xi_{12}\delta_2 < 0, \tag{3.25}$$

$$\delta_1 - \xi_{13}\delta_3 > 0, \quad \delta_2 - \xi_{23}\delta_3 < 0, \tag{3.26}$$

$$1 - \xi_{12}\xi_{21} \neq 0, \tag{3.27}$$

$$1 - \xi_{23}\xi_{32} \neq 0, \tag{3.28}$$

$$1 - \xi_{31}\xi_{13} \neq 0, \tag{3.29}$$

the slave system (3.4) has a heteroclinic cycle coinsisting of equilibrium points S_1, S_2, S_3 and heteroclinic trajectories $\underline{\Gamma_{12}}, \underline{\Gamma_{23}}, \underline{\Gamma_{31}}$.

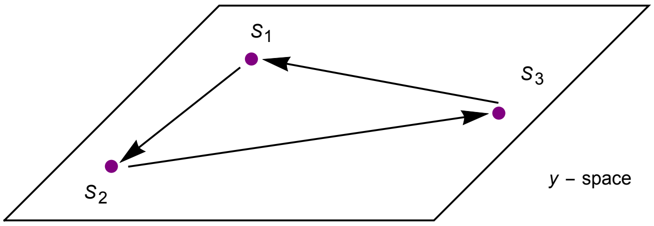

Figure 3.3: Heteroclinic cycle of slave system for $\eta_{1j} \equiv 0$.

Proof. The proof for existence of heteroclinic cycle in the slave system (3.4) when $\eta_{ij} \equiv 0$, is in exactly the same as for Proposition 3.2.1. ■

3.3 Conditions for the existence of "triangules" in a heteroclinic network

We shall study now the behavior at our system on invariant three-dimensional subspaces. We check directly that the plane $x_1 = \sigma_1, x_2 = 0, x_3 = 0$ is an invariant subspace and the system (3.4) restricted to this subspace has the form

$$\begin{cases} \dot{y_1} = y_1(\delta_1 - y_1 - \xi_{12}y_2 - \xi_{13}y_3 - \eta_{11}\sigma_1) \\ \dot{y_2} = y_2(\delta_2 - y_2 - \xi_{21}y_1 - \xi_{23}y_3 - \eta_{21}\sigma_1) \\ \dot{y_3} = y_3(\delta_3 - y_3 - \xi_{31}y_1 - \xi_{32}y_2 - \eta_{31}\sigma_1) \end{cases} \tag{3.30}$$

It follows that there are equilibrium points for it which are $\bar{O}_1 = (\delta_1 - \eta_{11}\ \sigma_1, 0\ , 0)$, $\bar{O}_2 = (0,\ \delta_2 - \eta_{21}\ \sigma_1, 0)$ and $\bar{O}_3 = (0\ ,\ 0,\ \delta_3 - \eta_{31}\ \sigma_1)$.

Proposition 3.3.1 *Assume that the coordinates of $O_1 = (\sigma_1, 0, 0)$ are sustituted to the slave system (3.4). Under the conditions (3.27)-(3.29) and*

$$-\delta_1+\eta_{11}\sigma_1 < 0, \delta_2-\eta_{21}\sigma_1-\xi_{21}(\delta_1-\eta_{11}\sigma_1) > 0, \delta_3-\eta_{31}\sigma_1-\xi_{31}(\delta_1-\eta_{11}\sigma_1) < 0, \tag{3.31}$$

$$-\delta_2+\eta_{21}\sigma_1 < 0, \delta_3-\eta_{31}\sigma_1-\xi_{32}(\delta_2-\eta_{21}\sigma_1) > 0, \delta_1-\eta_{11}\sigma_1-\xi_{12}(\delta_2-\eta_{21}\sigma_1) < 0, \tag{3.32}$$

$$-\delta_3+\eta_{31}\sigma_1 < 0, \delta_1-\eta_{11}\sigma_1-\xi_{13}(\delta_3-\eta_{31}\sigma_1) > 0, \delta_2-\eta_{21}\sigma_1-\xi_{23}(\delta_3-\eta_{31}\sigma_1) < 0, \tag{3.33}$$

the master-slave system (3.3),(3.4) has a heteroclinic cycle coinsisting of equilibrium points $\bar{O}_1, \bar{O}_2, \bar{O}_2$ and heteroclinic trajectories $\bar{\Gamma}_{12}, \bar{\Gamma}_{23}, \bar{\Gamma}_{31}$.

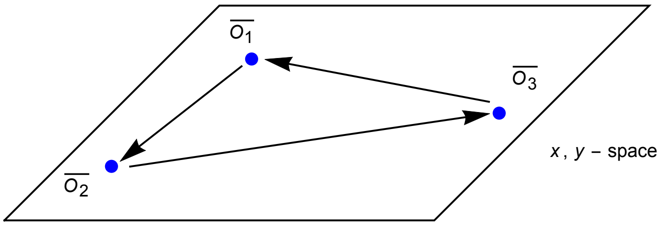

Figure 3.4: Heteroclinic cycle of master-slave system when the coordinates of O_1 are sustituted to the slave system.

Proof. If the coordinates of O_1 are sustituted to (3.4), we have the system (3.30) and the jacobian matrix for this system is

$$J_{\bar{y}_i} = \begin{pmatrix} \delta_1 - 2y_1 - \xi_{12}y_2 - \xi_{13}y_3 - \eta_{11}\sigma_1 & -\xi_{12}y_1 & -\xi_{13}y_1 \\ -\xi_{21}y_2 & \delta_2 - 2y_2 - \xi_{21}y_1 - \xi_{23}y_3 - \eta_{21}\sigma_1 & -\xi_{23}y_2 \\ -\xi_{31}y_3 & -\xi_{32}y_3 & \delta_3 - 2y_3 - \xi_{31}y_1 - \xi_{32}y_2 - \eta_{31}\sigma 1 \end{pmatrix} \tag{3.34}$$

For $\bar{O}_1$ the jacobian matrix is

$$J_{\bar{O}_1} = \begin{pmatrix} -\delta_1 + \eta_{11}\sigma_1 & -\xi_{12}(\delta_1 - \eta_{11}\sigma_1) & -\xi_{13}(\delta_1 - \eta_{11}\sigma_1) \\ 0 & \delta_2 - \eta_{21}\sigma_1 - \xi_{21}(\delta_1 - \eta_{11}\sigma_1) & 0 \\ 0 & 0 & \delta_3 - \eta_{31}\sigma_1 - \xi_{31}(\delta_1 - \eta_{11}\sigma_1) \end{pmatrix} \tag{3.35}$$

Thus the eigenvalues are $\lambda_1{}^{\bar{1}}$=-$\delta_1 + \eta_{11}\sigma_1$,$\lambda_2{}^{\bar{1}}= \delta_2 - \eta_{21}\sigma_1 - \xi_{21}(\delta_1 - \eta_{11}\sigma_1)$ and $\lambda_3{}^{\bar{1}}= \delta_3 - \eta_{31}\sigma_1 - \xi_{31}(\delta_1 - \eta_{11}\sigma_1)$. Because of (3.31) we have that $\bar{O}_1$ is a saddle equlilibrium point with one-dimensional unstable manifold.

For $\bar{O}_2$ the jacobian matrix is

$$J_{\bar{O}_2} = \begin{pmatrix} \delta_1 - \xi_{12}(\delta_2 - \eta_{21}\sigma_1) - \eta_{11}\sigma_1 & 0 & 0 \\ -\xi_{21}(\delta_2 - \eta_{21}\sigma_1) & -\delta_2 + \eta_{21}\sigma_1 & -\xi_{23}(\delta_2 - \eta_{21}\sigma_1) \\ 0 & 0 & \delta_3 - \eta_{31}\sigma_1 - \xi_{32}(\delta_2 - \eta_{21}\sigma_1) \end{pmatrix} \tag{3.36}$$

Thus the eigenvalues are $\lambda_1{}^{\bar{2}}=-\delta_2 + \eta_{21}\sigma_1$,$\lambda_2{}^{\bar{2}}= \delta_3 - \eta_{31}\sigma_1 - \xi_{32}(\delta_2 - \eta_{21}\sigma_1)$ and $\lambda_3{}^{\bar{2}}= \delta_1 - \xi_{12}(\delta_2 - \eta_{21}\sigma_1) - \eta_{11}\sigma_1$. Because of (3.32) we have that $\bar{O}_2$ is a saddle equlilibrium point with one-dimensional unstable manifold.

Finally for $\bar{O}_3$ the jacobian matrix is

$$J_{\bar{O}_3} = \begin{pmatrix} \delta_1 - \xi_{13}(\delta_3 - \eta_{31}\sigma_1) - \eta_{11}\sigma_1 & 0 & 0 \\ 0 & \delta_2 - \xi_{23}(\delta_3 - \eta_{31}\sigma_1) - \eta_{21}\sigma_1 & 0 \\ -\xi_{31}(\delta_3 - \eta_{31}\sigma_1) & -\xi_{32}(\delta_3 - \eta_{31}\sigma_1) & -\delta_3 + \eta_{31}\sigma_1 \end{pmatrix} \tag{3.37}$$

Thus the eigenvalues are $\lambda_1{}^{\bar{3}}=-\delta_3 + \eta_{31}\sigma_1$,$\lambda_2{}^{\bar{3}}= \delta_1 - \xi_{13}(\delta_3 - \eta_{31}\sigma_1) - \eta_{11}\sigma_1$ and $\lambda_3{}^{\bar{3}}= \delta_2 - \xi_{23}(\delta_3 - \eta_{31}\sigma_1) - \eta_{21}\sigma_1$. Now because of (3.33) we have that $\bar{O}_3$ is a saddle equlilibrium point with one-dimensional unstable manifold.

Consider the plane $P_{y_1y_2}, y_3 = 0$, then the restriction of system (3.30) on this plane is the following

$$\begin{cases} \dot{y_1} = y_1(\delta_1 - \eta_{11}\sigma_1 - y_1 - \xi_{12}y_2) \\ \dot{y_2} = y_2(\delta_2 - \eta_{21}\sigma_1 - y_2 - \xi_{21}y_1) \end{cases} \tag{3.38}$$

One can see that under (3.31),(3.32) and (3.27) are satisfaced the conditions of Proposition 3.1.2, it implies that there exists a heteroclinic trayectory $\bar{\Gamma_{12}}$ joining $\bar{O_1}$ and $\bar{O_2}$.

Consider the plane $P_{y_2y_3}, y_1 = 0$, then the restriction of system (3.30) on this plane is the following

$$\begin{cases} \dot{y_2} = y_2(\delta_2 - \eta_{21}\sigma_1 - y_2 - \xi_{23}y_3) \\ \dot{y_3} = y_3(\delta_3 - \eta_{31}\sigma_1 - y_3 - \xi_{32}y_2) \end{cases} \tag{3.39}$$

One can see that under (3.32),(3.33) and (3.28) are satisfaced the conditions of Proposition 3.1.2, it implies that there exists a heteroclinic trayectory $\bar{\Gamma_{23}}$ joining $\bar{O_2}$ and $\bar{O_3}$.

Finally consider $P_{y_3y_1}, y_2 = 0$, then the restriction of system (3.30) on this plane is the following

$$\begin{cases} \dot{y_3} = y_3(\delta_3 - \eta_{31}\sigma_1 - y_3 - \xi_{31}y_1) \\ \dot{y_1} = y_1(\delta_1 - \eta_{11}\sigma_1 - y_1 - \xi_{13}y_3) \end{cases} \tag{3.40}$$

Under (3.31),(3.33) and (3.29) are satisfaced the conditions of Proposition 3.1.2, it implies that there exists a heteroclinic trayectory $\bar{\Gamma_{31}}$ joining $\bar{O_3}$ and $\bar{O_1}$. So the set $\bar{\Gamma_3} = (\bar{O_1} \cup \bar{O_2} \cup \bar{O_3}) \cup (\bar{\Gamma_{12}} \cup \bar{\Gamma_{23}} \cup \bar{\Gamma_{31}})$ is a heteroclinic cycle for master-slave system. ■

Now we check directly that the plane $x_2 = \sigma_2, x_1 = 0, x_3 = 0$ is an invariant subspace and the system (3.4) restricted to this subspace has the form

$$\begin{cases} \dot{y_1} = y_1(\delta_1 - y_1 - \xi_{12}y_2 - \xi_{13}y_3 - \eta_{12}\sigma_2) \\ \dot{y_2} = y_2(\delta_2 - y_2 - \xi_{21}y_1 - \xi_{23}y_3 - \eta_{22}\sigma_2) \\ \dot{y_3} = y_3(\delta_3 - y_3 - \xi_{31}y_1 - \xi_{32}y_2 - \eta_{32}\sigma_2) \end{cases} \tag{3.41}$$

It follows that there are equilibrium points for it which are $\dot{O_1} = (\delta_1 - \eta_{12}\ \sigma_2, 0, 0)$, $\dot{O_2} = (0, \delta_2 - \eta_{22}\ \sigma_2, 0)$ and $\dot{O_3} = (0, 0, \delta_3 - \eta_{32}\ \sigma_2)$.

Proposition 3.3.2 *Assume that the coordinates of $O_2 = (0, \sigma_2, 0)$ are sustituted to the slave system (3.4). Under the conditions (3.27)-(3.29) and*

$$-\delta_1 + \eta_{12}\sigma_2 < 0, \delta_2 - \eta_{22}\sigma_2 - \xi_{21}(\delta_1 - \eta_{12}\sigma_2) > 0, \delta_3 - \eta_{32}\sigma_2 - \xi_{31}(\delta_1 - \eta_{12}\sigma_2) < 0, \tag{3.42}$$

$$-\delta_2 + \eta_{22}\sigma_2 < 0, \delta_3 - \eta_{32}\sigma_2 - \xi_{32}(\delta_2 - \eta_{22}\sigma_2) > 0, \delta_1 - \eta_{12}\sigma_2 - \xi_{12}(\delta_2 - \eta_{22}\sigma_2) < 0, \tag{3.43}$$

$$-\delta_3 + \eta_{32}\sigma_2 < 0, \delta_1 - \eta_{12}\sigma_2 - \xi_{13}(\delta_3 - \eta_{32}\sigma_2) > 0, \delta_2 - \eta_{22}\sigma_2 - \xi_{23}(\delta_3 - \eta_{32}\sigma_2) < 0, \tag{3.44}$$

the master-slave system (3.3),(3.4) has a heteroclinic cycle coinsisting of equilibrium points $\dot{O_1}, \dot{O_2}, \dot{O_3}$ and heteroclinic trajectories $\dot{\Gamma}_{12}, \dot{\Gamma}_{23}, \dot{\Gamma}_{31}$.

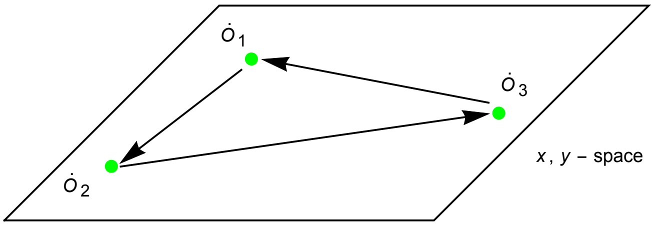

Figure 3.5: Heteroclinic cycle of master-slave system when the coordinates of O_2 are sustituted to slave system.

Proof. If the coordinates of O_2 are sustituted to (3.4), we have the system (3.41) and the proof is exactly the same as for Proposition 3.3.1. ■

Finally we check directly that the plane $x_3 = \sigma_3, x_1 = 0, x_2 = 0$ is an invariant subspace and the system (3.4) restricted to this subspace has the form

$$\begin{cases} \dot{y_1} = y_1(\delta_1 - y_1 - \xi_{12}y_2 - \xi_{13}y_3 - \eta_{13}\sigma_3) \\ \dot{y_2} = y_2(\delta_2 - y_2 - \xi_{21}y_1 - \xi_{23}y_3 - \eta_{23}\sigma_3) \\ \dot{y_3} = y_3(\delta_3 - y_3 - \xi_{31}y_1 - \xi_{32}y_2 - \eta_{33}\sigma_3) \end{cases} \tag{3.45}$$

It follows that there are equilibrium points for it which are $\hat{O}_1 = (\delta_1 - \eta_{13}\ \sigma_3, 0\ , 0)$, $\hat{O}_2 = (0,\ \delta_2 - \eta_{23}\ \sigma_3, 0)$ and $\hat{O}_3 = (0\ ,\ 0,\ \delta_3 - \eta_{33}\ \sigma_3)$.

Proposition 3.3.3 *Assume that the coordinates of $O_3 = (0, 0, \sigma_3)$ are sustituted to the slave system (3.4).Under the conditions (3.27)-(3.29) and*

$$-\delta_1+\eta_{13}\sigma_3 < 0, \delta_2-\eta_{23}\sigma_3-\xi_{21}(\delta_1-\eta_{13}\sigma_3) > 0, \delta_3-\eta_{33}\sigma_3-\xi_{31}(\delta_1-\eta_{13}\sigma_3) < 0, \tag{3.46}$$

$$-\delta_2+\eta_{23}\sigma_3 < 0, \delta_3-\eta_{33}\sigma_3-\xi_{32}(\delta_2-\eta_{23}\sigma_3) > 0, \delta_1-\eta_{13}\sigma_3-\xi_{12}(\delta_2-\eta_{23}\sigma_3) < 0, \tag{3.47}$$

$$-\delta_3+\eta_{33}\sigma_3 < 0, \delta_1-\eta_{13}\sigma_3-\xi_{13}(\delta_3-\eta_{33}\sigma_3) > 0, \delta_2-\eta_{23}\sigma_3-\xi_{23}(\delta_3-\eta_{33}\sigma_3) < 0, \tag{3.48}$$

the master-slave system (3.3),(3.4) has a heteroclinic cycle coinsisting of equilibrium points $\hat{O}_1, \hat{O}_2, \hat{O}_3$ and heteroclinic trajectories $\hat{\Gamma}_{12}, \hat{\Gamma}_{23}, \hat{\Gamma}_{31}$.

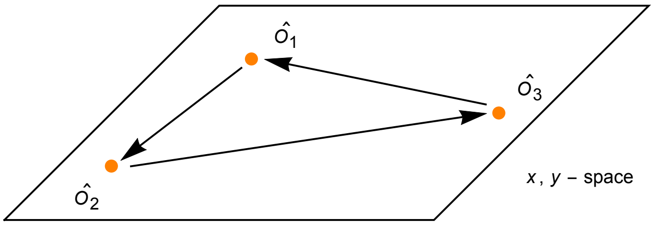

Figure 3.6: Heteroclinic cycle of master-slave system when the coordinates of O_3 are sustituted to slave system.

Proof If the coordinates of O_3 are sustituted to (3.4), we have the system (3.45) and the proof is exactly the same as for Proposition 3.3.1. ■

3.4 Study of the system in the full phase space

Obviously, we have the following nine equilibrium points for the system (3.3),(3.4)

$$\begin{cases} \bar{O_1} = (\sigma_1, 0, 0, \delta_1 - \eta_{11}\sigma_1, 0, 0), \\ \bar{O_2} = (\sigma_1, 0, 0, 0, \delta_2 - \eta_{21}\sigma_1, 0), \\ \bar{O_3} = (\sigma_1, 0, 0, 0, 0, \delta_3 - \eta_{31}\sigma_1). \end{cases} \tag{3.49}$$

$$\begin{cases} \dot{O_1} = (0, \sigma_2, 0, \delta_1 - \eta_{12}\sigma_2, 0, 0), \\ \dot{O_2} = (0, \sigma_2, 0, 0, \delta_2 - \eta_{22}\sigma_2, 0), \\ \dot{O_3} = (0, \sigma_2, 0, 0, 0, \delta_3 - \eta_{32}\sigma_2). \end{cases} \tag{3.50}$$

$$\begin{cases} \hat{O_1} = (0, 0, \sigma_3, \delta_1 - \eta_{13}\sigma_3, 0, 0), \\ \hat{O_2} = (0, 0, \sigma_3, 0, \delta_2 - \eta_{23}\sigma_3, 0), \\ \hat{O_3} = (0, 0, \sigma_3, 0, 0, \delta_3 - \eta_{33}\sigma_3). \end{cases} \tag{3.51}$$

Theorem 3.4.1 *Under the conditions (3.11)-(3.16),(3.27)-(3.29),(3.31)-(3.33),(3.42)-(3.44) and (3.46)-(3.48) the system (3.3),(3.4) has a heteroclinic network* Γ *containing points (3.49)-(3.51), as it is shown in the figure(3.7).*

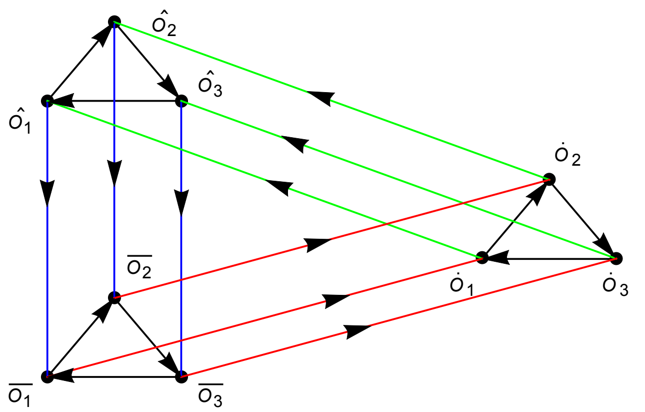

Figure 3.7: Heteroclinic network Γ of master-slave system.

Proof. Under the conditions (3.27)-(3.33),(3.42)-(3.44) and (3.46)-(3.48) the conditions of Proposition 3.3.1,Proposition 3.3.2 and Proposition 3.3.3 are satisfaced. Then the system (3.3),(3.4) has the heteroclinic cycles as it is shown in the Figure(3.8).

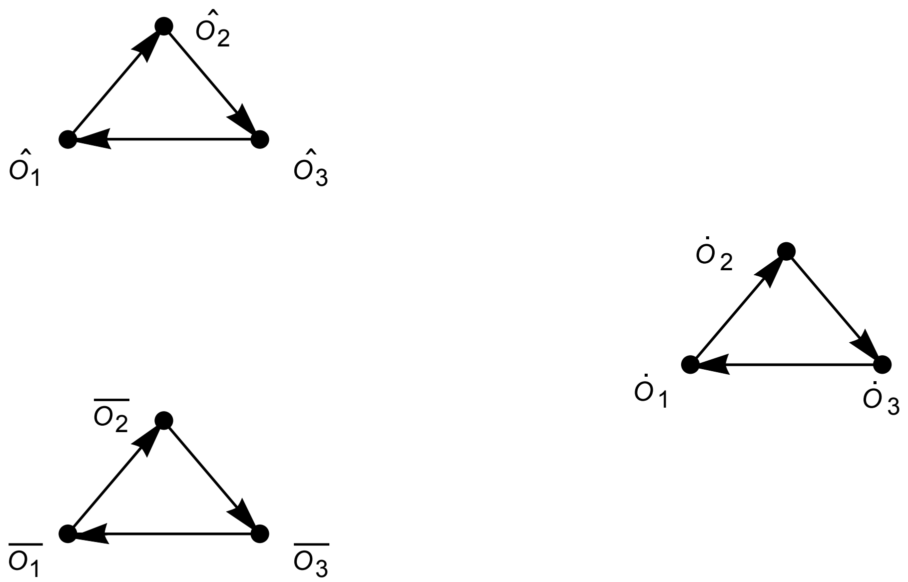

Figure 3.8: Heteroclinic cycles of master-slave system.

We need to show that
i)there exists a heteroclinic trajectory joining $\bar{O}_i$ and $\dot{O}_i$.
ii)there exists a heteroclinic trajectory joining $\dot{O}_i$ and $\hat{O}_i$.
iii)there exists a heteroclinic trajectory joining $\hat{O}_i$ and $\bar{O}_i$.

i) First let us prove that there exists a heteroclinic trajectory joining $\bar{O}_1$ and $\dot{O}_1$. These equilibrium points have coordinates $\bar{O}_1 = (\sigma_1, 0, 0, \delta_1 - \eta_{11}\sigma_1, 0, 0)$ and $\dot{O}_1 = (0, \sigma_2, 0, \delta_1 - \eta_{12}\sigma_2, 0, 0)$. Now consider in (3.3),(3.4) the 3-dimensional invariant subspace

$$x_3 = 0, y_2 = 0, y_3 = 0. \tag{3.52}$$

so we have the system

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{12}x_2) \\ \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{21}x_1) \\ \dot{y_1} = y_1(\delta_1 - y_1 - \eta_{11}x_1 - \eta_{12}x_2) \end{cases} \tag{3.53}$$

On this subspace $\bar{O}_1 = (\sigma_1, 0, \delta_1 - \eta_{11}\sigma_1)$ and $\dot{O}_1 = (0, \sigma_2, \delta_1 - \eta_{12}\sigma_2)$ are equilibrium points for system (3.53). Now jacobian matrix is the following

$$\begin{pmatrix} \sigma_1 - 2x_1 - \rho_{12}x_2 & -\rho_{12}x_1 & 0 \\ -\rho_{21}x_2 & \sigma_2 - \rho_{21}x_1 - 2x_2 & 0 \\ -\eta_{11}y_1 & -\eta_{12}y_1 & \delta_1 - \eta_{11}x_1 - \eta_{12}x_2 - 2y_1 \end{pmatrix} \tag{3.54}$$

For $\bar{O}_1$ we have

$$\begin{pmatrix} -\sigma_1 & -\rho_{12}\sigma_1 & 0 \\ 0 & \sigma_2 - \rho_{21}\sigma_1 & 0 \\ -\eta_{11}(\delta_1 - \eta_{11}\sigma_1) & -\eta_{12}(\delta_1 - \eta_{11}\sigma_1) & -\delta_1 + \eta_{11}\sigma_1 \end{pmatrix} \tag{3.55}$$

Thus eigenvalues are $\lambda_1(\bar{O}_1) = -\sigma_1, \lambda_2(\bar{O}_1) = \sigma_2 - \rho_{21}\sigma_1, \lambda_3(\bar{O}_1) = -\delta_1 + \eta_{11}\sigma_1$. This equilibrium point is a saddle because (3.11) and (3.31). The eigenvectors at $\bar{O}_1$ are

$$\begin{cases} v_1 = (-\frac{\sigma_1+\eta_{11}\sigma_1-\delta_1}{\eta_{11}(\eta_{11}\sigma_1-\delta_1)}, 0, 1), \\ v_2 = (\frac{\rho_{12}\sigma_1}{-\sigma_1+\rho_{21}\sigma_1-\sigma_2}, 1, -\frac{(-\eta_{12}\sigma_1+\eta_{11}\rho_{12}\sigma_1+\eta_{12}\rho_{21}\sigma_1-\eta_{12}\sigma_2)(\eta_{11}\sigma_1-\delta_1)}{(-\sigma_1+\rho_{21}\sigma_1-\sigma_2)(\eta_{11}\sigma_1+\rho_{21}\sigma_1-\sigma_2-\delta_1)}), \\ v_3 = (0, 0, 1). \end{cases} \tag{3.56}$$

where v_2 corresponds to $\lambda_2(\bar{O}_1) = \sigma_2 - \rho_{21}\sigma_1 > 0$. We see that the vector $(\frac{\rho_{12}\sigma_1}{-\sigma_1+\rho_{21}\sigma_1-\sigma_2}, 1)$ is an eigenvector of the master subsystem

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{12}x_2) \\ \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{21}x_1) \end{cases} \tag{3.57}$$

of the system (3.53) at the equilibrium point $W_1 = (\sigma_1, 0)$. Moreover the projection of any trajectory of the system (3.53) onto $P_{x_1x_2}$ is exactly a trajectory of the system (3.57).

Now for $\dot{O}_1$ we have

$$\begin{pmatrix} \sigma_1 - \rho_{12}\sigma_2 & 0 & 0 \\ -\rho_{21}\sigma_2 & -\sigma_2 & 0 \\ -\eta_{11}(\delta_1 - \eta_{12}\sigma_2) & -\eta_{12}(\delta_1 - \eta_{12}\sigma_2) & -\delta_1 + \eta_{12}\sigma_2 \end{pmatrix} \tag{3.58}$$

Thus eigenvalues are $\lambda_1(\dot{O}_1) = -\sigma_2, \lambda_2(\dot{O}_1) = \sigma_1 - \rho_{12}\sigma_2, \lambda_3(\dot{O}_1) = -\delta_1 + \eta_{12}\sigma_2$. This equilibrium point is a node because (3.12) and (3.42). We know that the projection of the unstable manifold of the point $\bar{O}_1$ onto the plane $y_1 = 0$ is the unstable manifold of the point W_1. We proved in Proposition 3.1.2 that the separatrix of W_1 goes to the equilibrium point $W_2 = (0, \sigma_2)$ (i.e) it is a heteroclinic trajectory. It implies that a separatrix of $\bar{O}_1$ goes to $\dot{O}_1$. Indeed it comes to be close to the invariant line $x_1 = 0, x_2 = \sigma_2, y_1 > 0$ and (see Proposition 3.4.1) since $\dot{O}_1$ is a global attractor on this line and an attractor of the system (3.53), it should go to $\dot{O}_1$.

Proposition 3.4.1 *The point $\dot{O}_1$ is a global attractor of the system (3.53) on the line $x_1 = 0, x_2 = \sigma_2, y_1 > 0$.*

Proof. We first show that $\dot{O}_1$ is an attractor. For that we have to check that eigenvalues of the matrix $L(\dot{O}_1)$ of the linearization of the system (3.53) at the point $\dot{O}_1$ have negative real parts. we see that $L(\dot{O}_1)$ coincide with (3.58) and it has eigenvalues $\lambda_i(\dot{O}_1) < 0, i = 1, 2, 3$ as was shown above. Second, the system (3.53) on this line has the form

$$\dot{y_1} = y_1(\delta_1 - y_1 - \eta_{12}\sigma_2). \tag{3.59}$$

Thus

$$\begin{cases} \dot{y_1} < 0 \text{ if } y_1 > \delta_1 - \eta_{12}\sigma_2 \text{ and} \\ \dot{y_1} > 0 \text{ if } 0 < y_1 < \delta_1 - \eta_{12}\sigma_2. \; \blacksquare \end{cases} \tag{3.60}$$

Heteroclinic connections existence between $\bar{O}_2$ and $\dot{O}_2$, $\bar{O}_3$ and $\dot{O}_3$ can be shown in the same way considering in (3.3),(3.4) the 3-dimensional invariant subspaces

$$x_3 = 0, y_1 = 0, y_3 = 0. \tag{3.61}$$

(i.e) the system

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{12}x_2) \\ \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{21}x_1) \\ \dot{y_2} = y_2(\delta_2 - y_2 - \eta_{11}x_1 - \eta_{12}x_2) \end{cases} \tag{3.62}$$

and

$$x_3 = 0, y_1 = 0, y_2 = 0. \tag{3.63}$$

(i.e) the system

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{12}x_2) \\ \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{21}x_1) \\ \dot{y_3} = y_3(\delta_3 - y_3 - \eta_{11}x_1 - \eta_{12}x_2) \end{cases} \tag{3.64}$$

respectively. Proof for *ii*) and *iii*) is in same way than above. Thus, we have proved the existence of heteroclinic trajectories $\bar{O}_i \to \dot{O}_i$, $\dot{O}_i \to \hat{O}_i$, $\hat{O}_i \to \bar{O}_i$, and hence, of a heteroclinic network Γ for the system (3.3)-(3.4). ■

3.5 Two-dimensional unstable manifolds

In this section we impose conditions under which all saddle equilibrium points in the heteroclinic network Γ will have two-dimensional unstable manifolds. Consider for instance, the point $\bar{O}_1$. The Jacobian matrix of system (3.3)-(3.4) has the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \tag{3.65}$$

where

$$A = \begin{pmatrix} \sigma_1 - 2x_1 - \rho_{12}x_2 - \rho_{13}x_3 & -\rho_{12}x_1 & -\rho_{13}x_3 \\ -\rho_{21}x_2 & \sigma_2 - 2x_2 - \rho_{21}x_1 - \rho_{23}x_3 & -\rho_{23}x_2 \\ -\rho_{31}x_3 & -\rho_{32}x_3 & \sigma_3 - 2x_3 - \rho_{31}x_1 - \rho_{32}x_2 \end{pmatrix},$$

$$B = \begin{pmatrix} -\eta_{11}y_1 & -\eta_{12}y_1 & -\eta_{13}y_1 \\ -\eta_{21}y_2 & -\eta_{22}y_2 & -\eta_{23}y_2 \\ -\eta_{31}y_3 & -\eta_{32}y_3 & -\eta_{33}y_3 \end{pmatrix},$$

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} c_1 & c_2 & c_3 \end{pmatrix},$$

with

$$c_1 = \begin{pmatrix} \delta_1 - \eta_{11}x_1 - \eta_{12}x_2 - \eta_{13}x_3 - 2y_1 - y_2\xi_{12} - y_3\xi_{13} \\ -y_2\xi_{21} \\ -y_3\xi_{31} \end{pmatrix},$$

$$c_2 = \begin{pmatrix} -y_1\xi_{12} \\ \delta_2 - \eta_{21}x_1 - \eta_{22}x_2 - \eta_{23}x_3 - y_1\xi_{21} - 2y_2 - y_3\xi_{23} \\ -y_3\xi_{32} \end{pmatrix},$$

$$c_3 = \begin{pmatrix} -y_1\xi_{13} \\ -y_2\xi_{23} \\ \delta_3 - \eta_{31}x_1 - \eta_{32}x_2 - \eta_{33}x_3 - y_1\xi_{31} - y_2\xi_{32} - 2y_3 \end{pmatrix}.$$

We can see that matrix (3.65) evaluated at $\bar{O}_1$ is a block diagonal matrix. So, its characteristic equation has the form of the product

$$m(\lambda)s(\lambda) = (\lambda^3 + a\lambda^2 + b\lambda + d)(\lambda^3 + e\lambda^2 + f\lambda + g) = 0 \tag{3.66}$$

Roots of the $m(\lambda)$ (master) are known. They are $\lambda_1 = -\sigma_1, \lambda_2 = \sigma_2 - \rho_{21}\sigma_1, \lambda_3 = \sigma_3 - \rho_{31}\sigma_1$; one of them is positive and two others are negative. Consider now the equation $s(\lambda)$ (slave). We obtain that

$$e = (\xi_{21} + \xi_{31} + 1)(\delta_1 - \eta_{11}\sigma_1) - \delta_2 - \delta_3 + \sigma_1(\eta_{21} + \eta_{31}\sigma_1), \tag{3.67}$$

$$f = \begin{cases} \delta_1^2(\xi_{21}\xi_{31} + \xi_{21} + \xi_{31}) - \delta_1(\delta_2(\xi_{31} + 1) + \delta_3(\xi_{21} + 1) - \sigma_1(-2\eta_{11}(\xi_{21}\xi_{31} + \xi_{21} \\ + \xi_{31}) + \eta_{21}(\xi_{31} + 1) + \eta_{31}(\xi_{21} + 1))) + \delta_2(\delta_3 + \sigma_1(\eta_{11}\xi_{31} + \eta_{11} - \eta_{31}))+ \\ \sigma_1\delta_3(\eta_{11}\xi_{21} + \eta_{11} - \eta_{21}) + \sigma_1(\eta_{11}^2(\xi_{21}\xi_{31} + \xi_{21} + \xi_{31}) - \eta_{11}(\eta_{21}\xi_{31} + \eta_{21}+ \\ \eta_{31}\xi_{21} + \eta_{31}) + \eta_{21}\eta_{31})), \end{cases} \tag{3.68}$$

and

$$g = (\delta_1 - \eta_{11}\sigma_1)(\xi_{21}(\delta_1 - \eta_{11}\sigma_1) - \delta_2 + \eta_{21}\sigma_1)(\xi_{31}(\delta_1 - \eta_{11}\sigma_1) - \delta_3 + \eta_{31}\sigma_1). \tag{3.69}$$

We use the following result (Proposition) of [6].

Proposition 3.5.1 *We consider the equation*

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0. \tag{3.70}$$

Let $\delta = pq - r$. If $r < 0, q \leq 0$ or $r < 0, q > 0, \delta > 0$ then, the roots $\lambda_1, \lambda_2, \lambda_3$ satisfy the inequalities

$$Re\lambda_1 < 0, Re\lambda_2 < 0, Re\lambda_3 > 0. \tag{3.71}$$

In our case $p = e$, $q = f$, $r = g$ and $\delta = \gamma$, where

$$\gamma = \begin{cases} (\delta_1(\xi_{21} + \xi_{31} + 1) - \delta_2 - \delta_3 - \eta_{11}\sigma_1\xi_{21} - \eta_{11}\sigma_1\xi_{31} - \eta_{11}\sigma_1 + \eta_{21}\sigma_1 + \eta_{31}\sigma_1) \\ (\delta_1^2(\xi_{21}\xi_{31} + \xi_{21} + \xi_{31}) - \delta_1(\delta_2(\xi_{31} + 1) + \delta_3(\xi_{21} + 1) - \sigma_1(-2\eta_{11}(\xi_{21}\xi_{31} + \xi_{21} \\ + \xi_{31}) + \eta_{21}(\xi_{31} + 1) + \eta_{31}(\xi_{21} + 1))) + \delta_2(\delta_3 + \sigma_1(\eta_{11}\xi_{31} + \eta_{11} - \eta_{31})) + \sigma_1 \\ (\delta_3(\eta_{11}\xi_{21} + \eta_{11} - \eta_{21}) + \sigma_1(\eta_{11}^2(\xi_{21}\xi_{31} + \xi_{21} + \xi_{31}) - \eta_{11}(\eta_{21}\xi_{31} + \eta_{21} + \eta_{31}\xi_{21} \\ + \eta_{31}) + \eta_{21}\eta_{31}))) - (\delta_1 - \eta_{11}\sigma_1)(\xi_{21}(\delta_1 - \eta_{11}\sigma_1) - \delta_2 + \eta_{21}\sigma_1)(\xi_{31}(\delta_1 - \eta_{11} \\ \sigma_1) - \delta_3 + \eta_{31}\sigma_1) \end{cases} \tag{3.72}$$

We obtain the following result

Lemma 3.5.1 *If the inequalities* $g < 0, f \leq 0$ *or* $g < 0, f > 0, \gamma > 0$ *are satisfied then,* $W^u(\bar{O}_1) = 2$.

Proof. First we know that the characteristic equation (3.66) of Jacobian matrix evaluated in $\bar{O}_1$ has the eigenvalues (roots of $m(\lambda)$) $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 < 0$. Second by hypothesis we have that $g < 0, f \leq 0$ or $g < 0, f > 0, \gamma > 0$. It implies (because of Proposition 3.5.1) that another eigenvalues (roots of $s(\lambda)$) λ_4, λ_5 and λ_6 satisfy the inequalities

$$Re\lambda_4 < 0, Re\lambda_5 < 0, Re\lambda_6 > 0.$$

So, there are two eigenvalues λ_2 and λ_6 such that $Re\lambda_{2,6} > 0$. The rest of eigenvalues $\lambda_{1,3,4,5}$, satisfy that $Re\lambda_{1,3,4,5} < 0$. It implies that $W^u(\bar{O}_1) = 2$. ■

Thus, the equilibrium point $\bar{O}_1$ has exactly 2 positive eigenvalues and 4 negative ones. Similarly we obtain

Lemma 3.5.2 *Consider* $O^\star \in O = \{\bar{O}_1, \bar{O}_2, \bar{O}_3, \dot{O}_1, \dot{O}_2, \dot{O}_3, \hat{O}_1, \hat{O}_2, \hat{O}_3\}$. *If the inequalities* $g < 0, f \leq 0$ *or* $g < 0, f > 0, \gamma > 0$*(where* f, g, γ *are the corresponding to* $O^\star$*, which are shown in the tables 3.1 and 3.2) are satisfied then,* $W^u(O^\star) = 2$.

Proof. The proof for $O^\star \in O$ is exactly the same as for Lemma 3.5.1. ■

We present in the tables (3.1) and (3.2) bellow the values of coefficients in characteristic polynomials in terms of parameters of the original system for all equilibrium points belonging to the heteroclinic network.

S	**C**	**Values of coefficients (C) of $s(\lambda)$ for every saddle (S)**
$\bar{O}_2$	e	$(\xi_{12}+\xi_{32}+1)(\delta_2-\eta_{21}\sigma_1)-\delta_1-\delta_3+\sigma_1(\eta_{11}+\eta_{31})$
	f	$\delta_2(\sigma_1(\eta_{31}(\xi_{12}+1)+\eta_{11}(\xi_{32}+1)-2\eta_{21}(\xi_{32}+\xi_{12}(\xi_{32}+1)))-\delta_3(\xi_{12}+1))+\delta_1(-\delta_2(\xi_{32}+1)+\delta_3+\sigma_1(\eta_{21}(\xi_{32}+1)-\eta_{31}))+\sigma_1(\delta_3(\eta_{21}(\xi_{12}+1)-\eta_{11})+\sigma_1(\eta_{21}(\eta_{21}(\xi_{32}+\xi_{12}(\xi_{32}+1))-\eta_{31}(\xi_{12}+1))-\eta_{11}(\eta_{21}(\xi_{32}+1)-\eta_{31})))+\delta_2^2(\xi_{32}+\xi_{12}(\xi_{32}+1))$
	g	$(\delta_2-\eta_{21}\sigma_1)(\delta_2\xi_{12}-\delta_1+\sigma_1(\eta_{11}-\eta_{21}\xi_{12}))(\delta_2\xi_{32}-\delta_3+\sigma_1(\eta_{31}-\eta_{21}\xi_{32}))$
	γ	$(\delta_2(\xi_{12}+\xi_{32}+1)-\delta_1-\delta_3-\eta_{21}\xi_{12}\sigma_1-\eta_{21}\xi_{32}\sigma_1+\eta_{11}\sigma_1-\eta_{21}\sigma_1+\eta_{31}\sigma_1)(\delta_2(\sigma_1(\eta_{31}(\xi_{12}+1)+\eta_{11}(\xi_{32}+1)-2\eta_{21}(\xi_{32}+\xi_{12}(\xi_{32}+1)))-\delta_3(\xi_{12}+1))+\delta_1(-\delta_2(\xi_{32}+1)+\delta_3+\sigma_1(\eta_{21}(\xi_{32}+1)-\eta_{31}))+\sigma_1(\delta_3(\eta_{21}(\xi_{12}+1)-\eta_{11})+\sigma_1(\eta_{21}(\eta_{21}(\xi_{32}+\xi_{12}(\xi_{32}+1))-\eta_{31}(\xi_{12}+1))-\eta_{11}(\eta_{21}(\xi_{32}+1)-\eta_{31})))+\delta_2^2(\xi_{32}+\xi_{12}(\xi_{32}+1)))-(\delta_2-\eta_{21}\sigma_1)(\delta_2\xi_{12}-\delta_1+\sigma_1(\eta_{11}-\eta_{21}\xi_{12}))(\delta_2\xi_{32}-\delta_3+\sigma_1(\eta_{31}-\eta_{21}\xi_{32}))$
$\bar{O}_3$	e	$(\xi_{13}+\xi_{23}+1)(\delta_3-\eta_{31}\sigma_1)-\delta_1-\delta_2+\sigma_1(\eta_{11}+\eta_{21})$
	f	$\delta_3\eta_{21}\xi_{13}\sigma_1-2\delta_3\eta_{31}\xi_{13}\sigma_1+\delta_3\eta_{11}\xi_{23}\sigma_1-2\delta_3\eta_{31}\xi_{23}\sigma_1-2\delta_3\eta_{31}\xi_{13}\xi_{23}\sigma_1-\delta_2(\delta_3(\xi_{13}+1)+\sigma_1(\eta_{11}-\eta_{31}(\xi_{13}+1)))+\delta_1(-\delta_3(\xi_{23}+1)+\delta_2+\sigma_1(\eta_{31}(\xi_{23}+1)-\eta_{21}))+\delta_3\eta_{11}\sigma_1+\delta_3\eta_{21}\sigma_1+\delta_3^2\xi_{13}+\delta_3^2\xi_{13}\xi_{23}+\delta_3^2\xi_{23}+\eta_{31}^2\xi_{13}\sigma_1^2-\eta_{21}\eta_{31}\xi_{13}\sigma_1^2+\eta_{31}^2\xi_{23}\sigma_1^2-\eta_{11}\eta_{31}\xi_{23}\sigma_1^2+\eta_{31}^2\xi_{13}\xi_{23}\sigma_1^2+\eta_{11}\eta_{21}\sigma_1^2-\eta_{11}\eta_{31}\sigma_1^2-\eta_{21}\eta_{31}\sigma_1^2$
	g	$(\delta_3-\eta_{31}\sigma_1)(-\delta_3\xi_{13}+\delta_1+\sigma_1(\eta_{31}\xi_{13}-\eta_{11}))(-\delta_3\xi_{23}+\delta_2+\sigma_1(\eta_{31}\xi_{23}-\eta_{21}))$
	γ	$(\delta_3\xi_{13}+\delta_3\xi_{23}-\delta_1-\delta_2+\delta_3-\eta_{31}\xi_{13}\sigma_1-\eta_{31}\xi_{23}\sigma_1+\eta_{11}\sigma_1+\eta_{21}\sigma_1-\eta_{31}\sigma_1)(\delta_3\eta_{21}\xi_{13}\sigma_1-2\delta_3\eta_{31}\xi_{13}\sigma_1+\delta_3\eta_{11}\xi_{23}\sigma_1-2\delta_3\eta_{31}\xi_{23}\sigma_1-2\delta_3\eta_{31}\xi_{13}\xi_{23}\sigma_1-\delta_2(\delta_3(\xi_{13}+1)+\sigma_1(\eta_{11}-\eta_{31}(\xi_{13}+1)))+\delta_1(-\delta_3(\xi_{23}+1)+\delta_2+\sigma_1(\eta_{31}(\xi_{23}+1)-\eta_{21}))+\delta_3\eta_{11}\sigma_1+\delta_3\eta_{21}\sigma_1+\delta_3^2\xi_{13}+\delta_3^2\xi_{13}\xi_{23}+\delta_3^2\xi_{23}+\eta_{31}^2\xi_{13}\sigma_1^2-\eta_{21}\eta_{31}\xi_{13}\sigma_1^2+\eta_{31}^2\xi_{23}\sigma_1^2-\eta_{11}\eta_{31}\xi_{23}\sigma_1^2+\eta_{31}^2\xi_{13}\xi_{23}\sigma_1^2+\eta_{11}\eta_{21}\sigma_1^2-\eta_{11}\eta_{31}\sigma_1^2-\eta_{21}\eta_{31}\sigma_1^2)-(\delta_3-\eta_{31}\sigma_1)(-\delta_3\xi_{13}+\delta_1+\sigma_1(\eta_{31}\xi_{13}-\eta_{11}))(-\delta_3\xi_{23}+\delta_2+\sigma_1(\eta_{31}\xi_{23}-\eta_{21}))$
$\dot{O}_1$	e	$(\xi_{21}+\xi_{31}+1)(\delta_1-\eta_{12}\sigma_2)-\delta_2-\delta_3+\sigma_2(\eta_{22}+\eta_{32})$
	f	$-\delta_1(\delta_3(\xi_{21}+1)+\delta_2(\xi_{31}+1)-\sigma_2(\eta_{32}(\xi_{21}+1)+\eta_{22}(\xi_{31}+1)-2\eta_{12}(\xi_{31}+\xi_{21}(\xi_{31}+1))))+\delta_2(\delta_3+\sigma_2(\eta_{12}(\xi_{31}+1)-\eta_{32}))+\sigma_2(\delta_3(\eta_{12}(\xi_{21}+1)-\eta_{22})+\sigma_2(\eta_{12}^2(\xi_{31}+\xi_{21}(\xi_{31}+1))-\eta_{12}(\eta_{32}(\xi_{21}+1)+\eta_{22}(\xi_{31}+1))+\eta_{22}\eta_{32}))+\delta_1^2(\xi_{31}+\xi_{21}(\xi_{31}+1))$
	g	$(\delta_1-\eta_{12}\sigma_2)(\delta_1\xi_{21}-\delta_2+\sigma_2(\eta_{22}-\eta_{12}\xi_{21}))(\delta_1\xi_{31}-\delta_3+\sigma_2(\eta_{32}-\eta_{12}\xi_{31}))$
	γ	$(\delta_1(\xi_{21}+\xi_{31}+1)-\delta_2-\delta_3-\eta_{12}\xi_{21}\sigma_2-\eta_{12}\xi_{31}\sigma_2-\eta_{12}\sigma_2+\eta_{22}\sigma_2+\eta_{32}\sigma_2)(-\delta_1(\delta_3(\xi_{21}+1)+\delta_2(\xi_{31}+1)-\sigma_2(\eta_{32}(\xi_{21}+1)+\eta_{22}(\xi_{31}+1)-2\eta_{12}(\xi_{31}+\xi_{21}(\xi_{31}+1))))+\delta_2(\delta_3+\sigma_2(\eta_{12}(\xi_{31}+1)-\eta_{32}))+\sigma_2(\delta_3(\eta_{12}(\xi_{21}+1)-\eta_{22})+\sigma_2(\eta_{12}^2(\xi_{31}+\xi_{21}(\xi_{31}+1))-\eta_{12}(\eta_{32}(\xi_{21}+1)+\eta_{22}(\xi_{31}+1))+\eta_{22}\eta_{32}))+\delta_1^2(\xi_{31}+\xi_{21}(\xi_{31}+1)))-(\delta_1-\eta_{12}\sigma_2)(\delta_1\xi_{21}-\delta_2+\sigma_2(\eta_{22}-\eta_{12}\xi_{21}))(\delta_1\xi_{31}-\delta_3+\sigma_2(\eta_{32}-\eta_{12}\xi_{31}))$
$\dot{O}_2$	e	$(\xi_{12}+\xi_{32}+1)(\delta_2-\eta_{22}\sigma_2)-\delta_1-\delta_3+\sigma_2(\eta_{12}+\eta_{32})$
	f	$\delta_2(\sigma_2(\eta_{32}(\xi_{12}+1)+\eta_{12}(\xi_{32}+1)-2\eta_{22}(\xi_{32}+\xi_{12}(\xi_{32}+1)))-\delta_3(\xi_{12}+1))+\delta_1(-\delta_2(\xi_{32}+1)+\delta_3+\sigma_2(\eta_{22}(\xi_{32}+1)-\eta_{32}))+\sigma_2(\delta_3(\eta_{22}(\xi_{12}+1)-\eta_{12})+\sigma_2(\eta_{22}(\eta_{22}(\xi_{32}+\xi_{12}(\xi_{32}+1))-\eta_{32}(\xi_{12}+1))-\eta_{12}(\eta_{22}(\xi_{32}+1)-\eta_{32})))+\delta_2^2(\xi_{32}+\xi_{12}(\xi_{32}+1))$
	g	$(\delta_2-\eta_{22}\sigma_2)(\delta_2\xi_{12}-\delta_1+\sigma_2(\eta_{12}-\eta_{22}\xi_{12}))(\delta_2\xi_{32}-\delta_3+\sigma_2(\eta_{32}-\eta_{22}\xi_{32}))$
	γ	$(\delta_2(\xi_{12}+\xi_{32}+1)-\delta_1-\delta_3-\eta_{22}\xi_{12}\sigma_2-\eta_{22}\xi_{32}\sigma_2+\eta_{12}\sigma_2-\eta_{22}\sigma_2+\eta_{32}\sigma_2)(\delta_2(\sigma_2(\eta_{32}(\xi_{12}+1)+\eta_{12}(\xi_{32}+1)-2\eta_{22}(\xi_{32}+\xi_{12}(\xi_{32}+1)))-\delta_3(\xi_{12}+1))+\delta_1(-\delta_2(\xi_{32}+1)+\delta_3+\sigma_2(\eta_{22}(\xi_{32}+1)-\eta_{32}))+\sigma_2(\delta_3(\eta_{22}(\xi_{12}+1)-\eta_{12})+\sigma_2(\eta_{22}(\eta_{22}(\xi_{32}+\xi_{12}(\xi_{32}+1))-\eta_{32}(\xi_{12}+1))-\eta_{12}(\eta_{22}(\xi_{32}+1)-\eta_{32})))+\delta_2^2(\xi_{32}+\xi_{12}(\xi_{32}+1)))-(\delta_2-\eta_{22}\sigma_2)(\delta_2\xi_{12}-\delta_1+\sigma_2(\eta_{12}-\eta_{22}\xi_{12}))(\delta_2\xi_{32}-\delta_3+\sigma_2(\eta_{32}-\eta_{22}\xi_{32}))$

Table 3.1: The table shows the values of coefficients of $s(\lambda)=\lambda^3+e\lambda^2+f\lambda+g$ and value of $\gamma=ef-g$ for saddles $\bar{O}_2,\bar{O}_3,\dot{O}_1$ and $\dot{O}_2$.

S	**C**	**Values of coefficients (C) of $s(\lambda)$ for every saddle (S)**
$\dot{O}_3$	e	$(\xi_{13}+\xi_{23}+1)(\delta_3-\eta_{32}\sigma_2)-\delta_1-\delta_2+\sigma_2(\eta_{12}+\eta_{22})$
	f	$\delta_3\eta_{22}\xi_{13}\sigma_2 - 2\delta_3\eta_{32}\xi_{13}\sigma_2 + \delta_3\eta_{12}\xi_{23}\sigma_2 - 2\delta_3\eta_{32}\xi_{23}\sigma_2 - 2\delta_3\eta_{32}\xi_{13}\xi_{23}\sigma_2 - \delta_2\left(\delta_3\left(\xi_{13}+1\right)+\sigma_2\left(\eta_{12}-\eta_{32}\left(\xi_{13}+1\right)\right)\right) + \delta_1\left(-\delta_3\left(\xi_{23}+1\right)+\delta_2+\sigma_2\left(\eta_{32}\left(\xi_{23}+1\right)-\eta_{22}\right)\right) + \delta_3\eta_{12}\sigma_2+\delta_3\eta_{22}\sigma_2+\delta_3^2\xi_{13}+\delta_3^2\xi_{13}\xi_{23}+\delta_3^2\xi_{23}+\eta_{32}^2\xi_{13}\sigma_2^2-\eta_{22}\eta_{32}\xi_{13}\sigma_2^2+\eta_{32}^2\xi_{23}\sigma_2^2-\eta_{12}\eta_{32}\xi_{23}\sigma_2^2+\eta_{32}^2\xi_{13}\xi_{23}\sigma_2^2+\eta_{12}\eta_{22}\sigma_2^2-\eta_{12}\eta_{32}\sigma_2^2-\eta_{22}\eta_{32}\sigma_2^2$
	g	$(\delta_3-\eta_{32}\sigma_2)\left(-\delta_3\xi_{13}+\delta_1+\sigma_2\left(\eta_{32}\xi_{13}-\eta_{12}\right)\right)\left(-\delta_3\xi_{23}+\delta_2+\sigma_2\left(\eta_{32}\xi_{23}-\eta_{22}\right)\right)$
	γ	$(\delta_3\xi_{13}+\delta_3\xi_{23}-\delta_1-\delta_2+\delta_3-\eta_{32}\xi_{13}\sigma_2-\eta_{32}\xi_{23}\sigma_2+\eta_{12}\sigma_2+\eta_{22}\sigma_2-\eta_{32}\sigma_2)(\delta_3\eta_{22}\xi_{13}\sigma_2-2\delta_3\eta_{32}\xi_{13}\sigma_2+\delta_3\eta_{12}\xi_{23}\sigma_2-2\delta_3\eta_{32}\xi_{23}\sigma_2-2\delta_3\eta_{32}\xi_{13}\xi_{23}\sigma_2-\delta_2(\delta_3(\xi_{13}+1)+\sigma_2(\eta_{12}-\eta_{32}(\xi_{13}+1)))+\delta_1(-\delta_3(\xi_{23}+1)+\delta_2+\sigma_2(\eta_{32}(\xi_{23}+1)-\eta_{22}))+\delta_3\eta_{12}\sigma_2+\delta_3\eta_{22}\sigma_2+\delta_3^2\xi_{13}+\delta_3^2\xi_{13}\xi_{23}+\delta_3^2\xi_{23}+\eta_{32}^2\xi_{13}\sigma_2^2-\eta_{22}\eta_{32}\xi_{13}\sigma_2^2+\eta_{32}^2\xi_{23}\sigma_2^2-\eta_{12}\eta_{32}\xi_{23}\sigma_2^2+\eta_{32}^2\xi_{13}\xi_{23}\sigma_2^2+\eta_{12}\eta_{22}\sigma_2^2-\eta_{12}\eta_{32}\sigma_2^2-\eta_{22}\eta_{32}\sigma_2^2)-(\delta_3-\eta_{32}\sigma_2)(-\delta_3\xi_{13}+\delta_1+\sigma_2(\eta_{32}\xi_{13}-\eta_{12}))(-\delta_3\xi_{23}+\delta_2+\sigma_2(\eta_{32}\xi_{23}-\eta_{22}))$
$\hat{O}_1$	e	$(\xi_{21}+\xi_{31}+1)(\delta_1-\eta_{13}\sigma_3)-\delta_2-\delta_3+\sigma_3(\eta_{23}+\eta_{33})$
	f	$-\delta_1(\delta_3(\xi_{21}+1)+\delta_2(\xi_{31}+1)-\sigma_3(\eta_{33}(\xi_{21}+1)+\eta_{23}(\xi_{31}+1)-2\eta_{13}(\xi_{31}+\xi_{21}(\xi_{31}+1))))+\delta_2(\delta_3+\sigma_3(\eta_{13}(\xi_{31}+1)-\eta_{33}))+\sigma_3(\delta_3(\eta_{13}(\xi_{21}+1)-\eta_{23})+\sigma_3(\eta_{13}^2(\xi_{31}+\xi_{21}(\xi_{31}+1))-\eta_{13}(\eta_{33}(\xi_{21}+1)+\eta_{23}(\xi_{31}+1))+\eta_{23}\eta_{33}))+\delta_1^2(\xi_{31}+\xi_{21}(\xi_{31}+1))$
	g	$(\delta_1-\eta_{13}\sigma_3)\left(\delta_1\xi_{21}-\delta_2+\sigma_3\left(\eta_{23}-\eta_{13}\xi_{21}\right)\right)\left(\delta_1\xi_{31}-\delta_3+\sigma_3\left(\eta_{33}-\eta_{13}\xi_{31}\right)\right)$
	γ	$(\delta_1(\xi_{21}+\xi_{31}+1)-\delta_2-\delta_3-\eta_{13}\xi_{21}\sigma_3-\eta_{13}\xi_{31}\sigma_3-\eta_{13}\sigma_3+\eta_{23}\sigma_3+\eta_{33}\sigma_3)(-\delta_1(\delta_3(\xi_{21}+1)+\delta_2(\xi_{31}+1)-\sigma_3(\eta_{33}(\xi_{21}+1)+\eta_{23}(\xi_{31}+1)-2\eta_{13}(\xi_{31}+\xi_{21}(\xi_{31}+1))))+\delta_2(\delta_3+\sigma_3(\eta_{13}(\xi_{31}+1)-\eta_{33}))+\sigma_3(\delta_3(\eta_{13}(\xi_{21}+1)-\eta_{23})+\sigma_3(\eta_{13}^2(\xi_{31}+\xi_{21}(\xi_{31}+1))-\eta_{13}(\eta_{33}(\xi_{21}+1)+\eta_{23}(\xi_{31}+1))+\eta_{23}\eta_{33}))+\delta_1^2(\xi_{31}+\xi_{21}(\xi_{31}+1)))-(\delta_1-\eta_{13}\sigma_3)(\delta_1\xi_{21}-\delta_2+\sigma_3(\eta_{23}-\eta_{13}\xi_{21}))(\delta_1\xi_{31}-\delta_3+\sigma_3(\eta_{33}-\eta_{13}\xi_{31}))$
$\hat{O}_2$	e	$(\xi_{12}+\xi_{32}+1)(\delta_2-\eta_{23}\sigma_3)-\delta_1-\delta_3+\sigma_3(\eta_{13}+\eta_{33})$
	f	$\delta_2(\sigma_3(\eta_{33}(\xi_{12}+1)+\eta_{13}(\xi_{32}+1)-2\eta_{23}(\xi_{32}+\xi_{12}(\xi_{32}+1)))-\delta_3(\xi_{12}+1))+\delta_1(-\delta_2(\xi_{32}+1)+\delta_3+\sigma_3(\eta_{23}(\xi_{32}+1)-\eta_{33}))+\sigma_3(\delta_3(\eta_{23}(\xi_{12}+1)-\eta_{13})+\sigma_3(\eta_{23}(\eta_{23}(\xi_{32}+\xi_{12}(\xi_{32}+1))-\eta_{33}(\xi_{12}+1))-\eta_{13}(\eta_{23}(\xi_{32}+1)-\eta_{33})))+\delta_2^2(\xi_{32}+\xi_{12}(\xi_{32}+1))$
	g	$(\delta_2-\eta_{23}\sigma_3)\left(\delta_2\xi_{12}-\delta_1+\sigma_3\left(\eta_{13}-\eta_{23}\xi_{12}\right)\right)\left(\delta_2\xi_{32}-\delta_3+\sigma_3\left(\eta_{33}-\eta_{23}\xi_{32}\right)\right)$
	γ	$(\delta_2(\xi_{12}+\xi_{32}+1)-\delta_1-\delta_3-\eta_{23}\xi_{12}\sigma_3-\eta_{23}\xi_{32}\sigma_3+\eta_{13}\sigma_3-\eta_{23}\sigma_3+\eta_{33}\sigma_3)(\delta_2(\sigma_3(\eta_{33}(\xi_{12}+1)+\eta_{13}(\xi_{32}+1)-2\eta_{23}(\xi_{32}+\xi_{12}(\xi_{32}+1)))-\delta_3(\xi_{12}+1))+\delta_1(-\delta_2(\xi_{32}+1)+\delta_3+\sigma_3(\eta_{23}(\xi_{32}+1)-\eta_{33}))+\sigma_3(\delta_3(\eta_{23}(\xi_{12}+1)-\eta_{13})+\sigma_3(\eta_{23}(\eta_{23}(\xi_{32}+\xi_{12}(\xi_{32}+1))-\eta_{33}(\xi_{12}+1))-\eta_{13}(\eta_{23}(\xi_{32}+1)-\eta_{33})))+\delta_2^2(\xi_{32}+\xi_{12}(\xi_{32}+1)))-(\delta_2-\eta_{23}\sigma_3)(\delta_2\xi_{12}-\delta_1+\sigma_3(\eta_{13}-\eta_{23}\xi_{12}))(\delta_2\xi_{32}-\delta_3+\sigma_3(\eta_{33}-\eta_{23}\xi_{32}))$
$\hat{O}_3$	e	$(\xi_{13}+\xi_{23}+1)(\delta_3-\eta_{33}\sigma_3)-\delta_1-\delta_2+\sigma_3(\eta_{13}+\eta_{23})$
	f	$\delta_3\eta_{23}\xi_{13}\sigma_3 - 2\delta_3\eta_{33}\xi_{13}\sigma_3 + \delta_3\eta_{13}\xi_{23}\sigma_3 - 2\delta_3\eta_{33}\xi_{23}\sigma_3 - 2\delta_3\eta_{33}\xi_{13}\xi_{23}\sigma_3 - \delta_2\left(\delta_3\left(\xi_{13}+1\right)+\sigma_3\left(\eta_{13}-\eta_{33}\left(\xi_{13}+1\right)\right)\right) + \delta_1\left(-\delta_3\left(\xi_{23}+1\right)+\delta_2+\sigma_3\left(\eta_{33}\left(\xi_{23}+1\right)-\eta_{23}\right)\right) + \delta_3\eta_{13}\sigma_3+\delta_3\eta_{23}\sigma_3+\delta_3^2\xi_{13}+\delta_3^2\xi_{13}\xi_{23}+\delta_3^2\xi_{23}+\eta_{33}^2\xi_{13}\sigma_3^2-\eta_{23}\eta_{33}\xi_{13}\sigma_3^2+\eta_{33}^2\xi_{23}\sigma_3^2-\eta_{13}\eta_{33}\xi_{23}\sigma_3^2+\eta_{33}^2\xi_{13}\xi_{23}\sigma_3^2+\eta_{13}\eta_{23}\sigma_3^2-\eta_{13}\eta_{33}\sigma_3^2-\eta_{23}\eta_{33}\sigma_3^2$
	g	$(\delta_3-\eta_{33}\sigma_3)\left(-\delta_3\xi_{13}+\delta_1+\sigma_3\left(\eta_{33}\xi_{13}-\eta_{13}\right)\right)\left(-\delta_3\xi_{23}+\delta_2+\sigma_3\left(\eta_{33}\xi_{23}-\eta_{23}\right)\right)$
	γ	$(\delta_3\xi_{13}+\delta_3\xi_{23}-\delta_1-\delta_2+\delta_3-\eta_{33}\xi_{13}\sigma_3-\eta_{33}\xi_{23}\sigma_3+\eta_{13}\sigma_3+\eta_{23}\sigma_3-\eta_{33}\sigma_3)(\delta_3\eta_{23}\xi_{13}\sigma_3-2\delta_3\eta_{33}\xi_{13}\sigma_3+\delta_3\eta_{13}\xi_{23}\sigma_3-2\delta_3\eta_{33}\xi_{23}\sigma_3-2\delta_3\eta_{33}\xi_{13}\xi_{23}\sigma_3-\delta_2(\delta_3(\xi_{13}+1)+\sigma_3(\eta_{13}-\eta_{33}(\xi_{13}+1)))+\delta_1(-\delta_3(\xi_{23}+1)+\delta_2+\sigma_3(\eta_{33}(\xi_{23}+1)-\eta_{23}))+\delta_3\eta_{13}\sigma_3+\delta_3\eta_{23}\sigma_3+\delta_3^2\xi_{13}+\delta_3^2\xi_{13}\xi_{23}+\delta_3^2\xi_{23}+\eta_{33}^2\xi_{13}\sigma_3^2-\eta_{23}\eta_{33}\xi_{13}\sigma_3^2+\eta_{33}^2\xi_{23}\sigma_3^2-\eta_{13}\eta_{33}\xi_{23}\sigma_3^2+\eta_{33}^2\xi_{13}\xi_{23}\sigma_3^2+\eta_{13}\eta_{23}\sigma_3^2-\eta_{13}\eta_{33}\sigma_3^2-\eta_{23}\eta_{33}\sigma_3^2)-(\delta_3-\eta_{33}\sigma_3)(-\delta_3\xi_{13}+\delta_1+\sigma_3(\eta_{33}\xi_{13}-\eta_{13}))(-\delta_3\xi_{23}+\delta_2+\sigma_3(\eta_{33}\xi_{23}-\eta_{23}))$

Table 3.2: The table shows the values of coefficients of $s(\lambda) = \lambda^3 + e\lambda^2 + f\lambda + g$ and value of $\gamma = ef - g$ for saddles $\dot{O}_3, \hat{O}_1, \hat{O}_2$ and $\hat{O}_3$.

Each heteroclinic trajectory belonging to the heteroclinic network lies in an invariant subspace of our system. It is important to obtain some information about behavior of the system in these subspaces.

3.6 Stability on 4-dimensional subspaces

We remind, first, the known stability criterion.

Let $(a_0, ...a_n)$ be the coefficients of the polynomial

$$a_0\lambda^n + a_1\lambda^{n-1} + ... + a_n. \tag{3.73}$$

We construct an $(n\mathrm{x}n)$-matrix

$$\tilde{A} = \begin{pmatrix} a_1 & a_3 & a_5 & ... & 0 & 0 \\ a_0 & a_2 & a_4 & ... & 0 & 0 \\ 0 & a_1 & a_3 & ... & 0 & 0 \\ 0 & a_0 & a_2 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & a_{n-2} & a_n \end{pmatrix} \tag{3.74}$$

and find the special minors $\tilde{\Delta}_i$, $i = 1, ..., n$, i.e. $\tilde{\Delta}_i$ is the determinant of the matrix whose entries lie on the intersection of the first i rows and the first i columns of matrix $\tilde{A}$.

Proposition 3.6.1 ***(Routh-Hurwitz criterion.)*** *All characteristic exponents have negative real parts if and only if each $\tilde{\Delta}_i$ is positive(see [5]).*

We apply now this criterion for all points in the heteroclinic trajectory starting with $\bar{O}_1$ in the invariant subspace $x_2 = y_2 = 0$.

Proposition 3.6.2 *Consider the invariant subspace of the system (3.3)-(3.4)*

$$x_2 = y_2 = 0. \tag{3.75}$$

If $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 are positive with

$$\Delta_1 = \xi_{31}(\delta_1 - \eta_{11}\sigma_1) + \delta_1 - \delta_3 - \eta_{11}\sigma_1 + \eta_{31}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 - \sigma_3,$$

$$\Delta_2 = \begin{cases} \delta_1^2(-\delta_3(2\xi_{31}+1) + \sigma_1(\eta_{31}(2\xi_{31}+1) - (\xi_{31}+1)(\xi_{31}(3\eta_{11} - \rho_{31} - 1) - \rho_{31} \\ -1)) + (\xi_{31}+1)^2(-\sigma_3)) + \delta_1(2\delta_3((\xi_{31}+1)\sigma_3 - \sigma_1(-\eta_{11}(2\xi_{31}+1) + \eta_{31} \\ +(\xi_{31}+1)(\rho_{31}+1))) + \delta_3^2 + \sigma_1^2(-2\eta_{11}(\eta_{31}(2\xi_{31}+1) + (\xi_{31}+1)^2(\rho_{31}+1)) \\ +2\eta_{31}(\xi_{31}+1)(\rho_{31}+1) + 3\eta_{11}^2\xi_{31}(\xi_{31}+1) + \eta_{31}^2 + (\xi_{31}+1)(\rho_{31}+1)^2) + 2 \\ (\xi_{31}+1)\sigma_1\sigma_3(\eta_{11}(\xi_{31}+1) - \eta_{31} - \rho_{31} - 1) + (\xi_{31}+1)\sigma_3^2) - \delta_3(\sigma_1(\eta_{11} - \rho_{31} \\ -1) + \sigma_3)(\sigma_1(\eta_{11}(2\xi_{31}+1) - 2\eta_{31} - \rho_{31} - 1) + \sigma_3) + \delta_3^2(\sigma_1(-\eta_{11} + \rho_{31} + 1) \\ -\sigma_3) + \delta_1^3\xi_{31}(\xi_{31}+1) + \sigma_1(\eta_{11}^2\sigma_1(\sigma_1(\eta_{31}(2\xi_{31}+1) + (\xi_{31}+1)^2(\rho_{31}+1)) - \\ (\xi_{31}+1)^2\sigma_3) - \eta_{11}(\sigma_1^2(2\eta_{31}(\xi_{31}+1)(\rho_{31}+1) + \eta_{31}^2 + (\xi_{31}+1)(\rho_{31}+1)^2) - \\ 2(\xi_{31}+1)\sigma_3\sigma_1(\eta_{31} + \rho_{31} + 1) + (\xi_{31}+1)\sigma_3^2) + \eta_{11}^3(-\xi_{31})(\xi_{31}+1)\sigma_1^2 + (\eta_{31} \\ +1)((\rho_{31}+1)\sigma_1 - \sigma_3)(\eta_{31}\sigma_1 + \rho_{31}\sigma_1 - \sigma_3)), \end{cases}$$

$$\Delta_3 = \begin{cases} (\xi_{31}((\rho_{31}+1)\sigma_1-\sigma_3)\delta_1^2+(\sigma_1((\xi_{31}+1)(\rho_{31}\sigma_1-\sigma_3)+\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3) \\ -2\eta_{11}\xi_{31}((\rho_{31}+1)\sigma_1-\sigma_3))-\delta_3((\rho_{31}+1)\sigma_1-\sigma_3))\delta_1+\sigma_1(\sigma_1(\xi_{31}((\rho_{31}+1) \\ \sigma_1-\sigma_3)\eta_{11}^2+(-(\xi_{31}+1)(\rho_{31}\sigma_1-\sigma_3)-\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3))\eta_{11}+\eta_{31}(\rho_{31} \\ \sigma_1-\sigma_3))+\delta_3(-\rho_{31}\sigma_1+\eta_{11}((\rho_{31}+1)\sigma_1-\sigma_3)+\sigma_3)))(\eta_{11}\eta_{31}\sigma_1^3+\eta_{11}\rho_{31}\sigma_1^3+ \\ \eta_{11}\eta_{31}\rho_{31}\sigma_1^3-\eta_{31}\rho_{31}\sigma_1^3-\delta_3\eta_{11}\sigma_1^2-\delta_1\eta_{31}\sigma_1^2-\delta_1\rho_{31}\sigma_1^2+\delta_3\rho_{31}\sigma_1^2-\delta_3\eta_{11}\rho_{31}\sigma_1^2 \\ -\delta_1\eta_{31}\rho_{31}\sigma_1^2+\eta_{11}\xi_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_1^2+\eta_{11}\xi_{31}\rho_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_1^2+\xi_{31}\rho_{31}(\eta_{11}\sigma_1 \\ -\delta_1)\sigma_1^2-\eta_{11}\sigma_3\sigma_1^2-\eta_{11}\eta_{31}\sigma_3\sigma_1^2+\eta_{31}\sigma_3\sigma_1^2+\delta_1\delta_3\sigma_1+\delta_1\delta_3\rho_{31}\sigma_1-\delta_1\xi_{31}(\delta_1- \\ \eta_{11}\sigma_1)\sigma_1-\delta_1\xi_{31}\rho_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_1+\delta_1\sigma_3\sigma_1-\delta_3\sigma_3\sigma_1+\delta_3\eta_{11}\sigma_3\sigma_1+\delta_1\eta_{31}\sigma_3\sigma_1 \\ +\xi_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_3\sigma_1+\eta_{11}\xi_{31}(\eta_{11}\sigma_1-\delta_1)\sigma_3\sigma_1-\delta_1\delta_3\sigma_3+\delta_1\xi_{31}(\delta_1-\eta_{11}\sigma_1) \\ \sigma_3+(\delta_1-\delta_3-\eta_{11}\sigma_1+\eta_{31}\sigma_1+\rho_{31}\sigma_1+\sigma_1+\xi_{31}(\delta_1-\eta_{11}\sigma_1)-\sigma_3)(\xi_{31}\delta_1^2+ \\ (-\delta_3+(\eta_{31}+\rho_{31}+\xi_{31}(-2\eta_{11}+\rho_{31}+1)+1)\sigma_1-(\xi_{31}+1)\sigma_3)\delta_1+\delta_3((\eta_{11}- \\ \rho_{31}-1)\sigma_1+\sigma_3)+\sigma_1(\xi_{31}\sigma_1\eta_{11}^2-((\eta_{31}+(\xi_{31}+1)(\rho_{31}+1))\sigma_1-(\xi_{31}+1)\sigma_3) \\ \eta_{11}+\rho_{31}\sigma_1+\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3)-\sigma_3)))-\sigma_1(\delta_1-\eta_{11}\sigma_1)(-\delta_3+\delta_1\xi_{31}+ \\ (\eta_{31}-\eta_{11}\xi_{31})\sigma_1)(\rho_{31}\sigma_1-\sigma_3)(\delta_3-\delta_1(\xi_{31}+1)+\eta_{11}\sigma_1-\eta_{31}\sigma_1+\eta_{11}\xi_{31}\sigma_1- \\ \rho_{31}\sigma_1-\sigma_1+\sigma_3)^2 \end{cases}$$

and

$$\Delta_4 = \begin{cases} \sigma_1(\delta_1-\eta_{11}\sigma_1)(-\delta_3+\delta_1\xi_{31}+(\eta_{31}-\eta_{11}\xi_{31})\sigma_1)(\rho_{31}\sigma_1-\sigma_3)((\xi_{31}((\rho_{31}+1)\sigma_1- \\ \sigma_3)\delta_1^2+(\sigma_1((\xi_{31}+1)(\rho_{31}\sigma_1-\sigma_3)+\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3)-2\eta_{11}\xi_{31}((\rho_{31}+1) \\ \sigma_1-\sigma_3))-\delta_3((\rho_{31}+1)\sigma_1-\sigma_3))\delta_1+\sigma_1(\sigma_1(\xi_{31}((\rho_{31}+1)\sigma_1-\sigma_3)\eta_{11}^2+(- \\ (\xi_{31}+1)(\rho_{31}\sigma_1-\sigma_3)-\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3))\eta_{11}+\eta_{31}(\rho_{31}\sigma_1-\sigma_3))+\delta_3(- \\ \rho_{31}\sigma_1+\eta_{11}((\rho_{31}+1)\sigma_1-\sigma_3)+\sigma_3)))(\eta_{11}\eta_{31}\sigma_1^3+\eta_{11}\rho_{31}\sigma_1^3+\eta_{11}\eta_{31}\rho_{31}\sigma_1^3-\eta_{31} \\ \rho_{31}\sigma_1^3-\delta_3\eta_{11}\sigma_1^2-\delta_1\eta_{31}\sigma_1^2-\delta_1\rho_{31}\sigma_1^2+\delta_3\rho_{31}\sigma_1^2-\delta_3\eta_{11}\rho_{31}\sigma_1^2-\delta_1\eta_{31}\rho_{31}\sigma_1^2+\eta_{11} \\ \xi_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_1^2+\eta_{11}\xi_{31}\rho_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_1^2+\xi_{31}\rho_{31}(\eta_{11}\sigma_1-\delta_1)\sigma_1^2-\eta_{11}\sigma_3\sigma_1^2 \\ -\eta_{11}\eta_{31}\sigma_3\sigma_1^2+\eta_{31}\sigma_3\sigma_1^2+\delta_1\delta_3\sigma_1+\delta_1\delta_3\rho_{31}\sigma_1-\delta_1\xi_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_1-\delta_1\xi_{31}\rho_{31} \\ (\delta_1-\eta_{11}\sigma_1)\sigma_1+\delta_1\sigma_3\sigma_1-\delta_3\sigma_3\sigma_1+\delta_3\eta_{11}\sigma_3\sigma_1+\delta_1\eta_{31}\sigma_3\sigma_1+\xi_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_3\sigma_1 \\ +\eta_{11}\xi_{31}(\eta_{11}\sigma_1-\delta_1)\sigma_3\sigma_1-\delta_1\delta_3\sigma_3+\delta_1\xi_{31}(\delta_1-\eta_{11}\sigma_1)\sigma_3+(\delta_1-\delta_3-\eta_{11}\sigma_1+ \\ \eta_{31}\sigma_1+\rho_{31}\sigma_1+\sigma_1+\xi_{31}(\delta_1-\eta_{11}\sigma_1)-\sigma_3)(\xi_{31}\delta_1^2+(-\delta_3+(\eta_{31}+\rho_{31}+\xi_{31}(-2 \\ \eta_{11}+\rho_{31}+1)+1)\sigma_1-(\xi_{31}+1)\sigma_3)\delta_1+\delta_3((\eta_{11}-\rho_{31}-1)\sigma_1+\sigma_3)+\sigma_1(\xi_{31}\sigma_1 \\ \eta_{11}^2-((\eta_{31}+(\xi_{31}+1)(\rho_{31}+1))\sigma_1-(\xi_{31}+1)\sigma_3)\eta_{11}+\rho_{31}\sigma_1+\eta_{31}((\rho_{31}+1) \\ \sigma_1-\sigma_3)-\sigma_3)))-\sigma_1(\delta_1-\eta_{11}\sigma_1)(-\delta_3+\delta_1\xi_{31}+(\eta_{31}-\eta_{11}\xi_{31})\sigma_1)(\rho_{31}\sigma_1-\sigma_3) \\ (\delta_3-\delta_1(\xi_{31}+1)+\eta_{11}\sigma_1-\eta_{31}\sigma_1+\eta_{11}\xi_{31}\sigma_1-\rho_{31}\sigma_1-\sigma_1+\sigma_3)^2). \end{cases}$$

Then the equilibrium point $\bar{O}_1$ is stable node on the invariant 4-dimensional subspace (3.75).

Proof. First the invariant 4-dimensional subspace (3.75) has the form

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{13}x_3) \\ \dot{x_3} = x_3(\sigma_3 - x_3 - \rho_{31}x_1) \\ \dot{y_1} = y_1(\delta_1 - y_1 - \xi_{13}y_3 - \eta_{11}x_1 - \eta_{13}x_3) \\ \dot{y_3} = y_3(\delta_3 - y_3 - \xi_{31}y_1 - \eta_{31}x_1 - \eta_{33}x_3). \end{cases} \tag{3.76}$$

We obtain that characteristic equation $p(\lambda)$ of Jacobian matrix of system (3.76) evaluated at equilibrium point $\bar{O_1}$ has the form

$$p(\lambda) = a_0\lambda^4 + a_1\lambda^3 a_2\lambda^2 + a_3\lambda + a_4, \tag{3.77}$$

where $a_0 = 1$, $a_1 = \xi_{31}(\delta_1 - \eta_{11}\sigma_1) + \delta_1 - \delta_3 - \eta_{11}\sigma_1 + \eta_{31}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 - \sigma_3$,

$$a_2 = \begin{cases} \delta_1(-\delta_3+\sigma_1(\xi_{31}(-2\eta_{11}+\rho_{31}+1)+\eta_{31}+\rho_{31}+1)-(\xi_{31}+1)\sigma_3)+\delta_3(\sigma_1 \\ (\eta_{11}-\rho_{31}-1)+\sigma_3)+\delta_1^2\xi_{31}+\sigma_1(-\eta_{11}(\sigma_1(\eta_{31}+(\xi_{31}+1)(\rho_{31}+1))-(\xi_{31} \\ +1)\sigma_3)+\eta_{11}^2\xi_{31}\sigma_1+\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3)+\rho_{31}\sigma_1-\sigma_3), \end{cases}$$

$$a_3 = \begin{cases} -\delta_1(\sigma_1(-2\eta_{11}\xi_{31}((\rho_{31}+1)\sigma_1-\sigma_3)+\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3)+(\xi_{31}+1)(\rho_{31} \\ \sigma_1-\sigma_3))-\delta_3((\rho_{31}+1)\sigma_1-\sigma_3))+\sigma_1(\delta_3(\eta_{11}((\rho_{31}+1)\sigma_1-\sigma_3)-\rho_{31}\sigma_1+\sigma_3) \\ +\sigma_1(\eta_{11}^2\xi_{31}((\rho_{31}+1)\sigma_1-\sigma_3)+\eta_{11}(-\eta_{31}((\rho_{31}+1)\sigma_1-\sigma_3)-(\xi_{31}+1)(\rho_{31}\sigma_1 \\ -\sigma_3))+\eta_{31}(\rho_{31}\sigma_1-\sigma_3)))+\delta_1^2\xi_{31}((\rho_{31}+1)\sigma_1-\sigma_3) \end{cases}$$

and
$a_4 = \sigma_1\left(\rho_{31}\sigma_1-\sigma_3\right)\left(\delta_1-\eta_{11}\sigma_1\right)\left(\delta_1\xi_{31}-\delta_3+\sigma_1\left(\eta_{31}-\eta_{11}\xi_{31}\right)\right).$

Now we construct the 4x4 matrix

$$\tilde{A} = \begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{pmatrix} \tag{3.78}$$

and find the special minors $\tilde{\Delta_1}, \tilde{\Delta_2}, \tilde{\Delta_3}$ and $\tilde{\Delta_4}$. One can check that $\Delta_1 = \tilde{\Delta_1}, \Delta_2 = \tilde{\Delta_2}, \Delta_3 = \tilde{\Delta_3}$ and $\Delta_4 = \tilde{\Delta_4}$. We know by hipotesis that all special minors are positive, therefore because of Routh-Hurwitz criterion the roots λ_i of (3.77) satisfy that $Re\lambda_{1,2,3,4} < 0$, and so $\bar{O_1}$ is stable node. ■

Similarly we obtain the following Proposition.

Proposition 3.6.3 *If $\Delta_1, \Delta_2, \Delta_3$ and Δ_4(where the special minors Δ_i are the corresponding to each equilibrium point, which are shown in tables 3.3,3.4,3.5 and 3.6) are positive, then the equilibrium points $\dot{O_2}, \dot{O_3}, \dot{O_1}, \hat{O_2}, \hat{O_3}, \hat{O_1}, \bar{O_2}$ and $\bar{O_3}$ are stable nodes on the invariant 4-dimensional subspaces of the system (3.3)-(3.4) $x_3 = y_3 = 0, x_3 = y_1 = 0, x_3 = y_2 = 0, x_1 = y_3 = 0, x_1 = y_1 = 0, x_1 = y_2 = 0, x_2 = y_3 = 0$ and $x_2 = y_1 = 0$ respectively.*

Proof. The proof for each one of equilibrium points is exactly the same as for Proposition 3.6.2. ■

The values of special minors in terms of parameters of the original system are shown in tables (3.3)-(3.6) bellow.

S	Δ_i	**Values of Δ_i for the equilibrium points $\dot{O}_2$ and $\dot{O}_3$**
$\dot{O}_2$	Δ_1	$\xi_{12}(\delta_2-\eta_{22}\sigma_2)-\delta_1+\delta_2+\eta_{12}\sigma_2-\eta_{22}\sigma_2+\rho_{12}\sigma_2-\sigma_1+\sigma_2$
	Δ_2	$\delta_2^2(\sigma_2(\eta_{12}(2\xi_{12}+1)-(\xi_{12}+1)(\xi_{12}(3\eta_{22}-\rho_{12}-1)-\rho_{12}-1))-(\xi_{12}+1)^2\sigma_1)+\delta_2(-2(\xi_{12}+1)\sigma_2\sigma_1(-\eta_{22}(\xi_{12}+1)+\eta_{12}+\rho_{12}+1)+\sigma_2^2(2\eta_{12}((\xi_{12}+1)(\rho_{12}+1)-\eta_{22}(2\xi_{12}+1))+(\xi_{12}+1)(-2\eta_{22}(\xi_{12}+1)(\rho_{12}+1)+3\eta_{22}^2\xi_{12}+(\rho_{12}+1)^2)+\eta_{12}^2)+(\xi_{12}+1)\sigma_1^2)-\delta_1(\delta_2+\sigma_2(-\eta_{22}+\rho_{12}+1)-\sigma_1)(\delta_2(2\xi_{12}+1)+\sigma_2(-\eta_{22}(2\xi_{12}+1)+2\eta_{12}+\rho_{12}+1)-\sigma_1)+\delta_1^2(\delta_2+\sigma_2(-\eta_{22}+\rho_{12}+1)-\sigma_1)+\delta_2^3\xi_{12}(\xi_{12}+1)+\sigma_2(-\sigma_2\sigma_1(-\eta_{22}(\xi_{12}+1)+\eta_{12}+1)(-\eta_{22}(\xi_{12}+1)+\eta_{12}+2\rho_{12}+1t)+\sigma_2^2(\eta_{22}^2(\xi_{12}+1)^2(\rho_{12}+1)-\eta_{22}(\xi_{12}+1)(\rho_{12}+1)^2+\eta_{12}(\eta_{22}-\rho_{12}-1)(\eta_{22}(2\xi_{12}+1)-\rho_{12}-1)+\eta_{22}^3(-\xi_{12})(\xi_{12}+1)+\eta_{12}^2(-\eta_{22}+\rho_{12}+1)+\rho_{12}(\rho_{12}+1))+\sigma_1^2(-\eta_{22}(\xi_{12}+1)+\eta_{12}+1))$
	Δ_3	$(\xi_{12}((\rho_{12}+1)\sigma_2-\sigma_1)\delta_2^2+\sigma_2(((\xi_{12}+1)\rho_{12}+\eta_{12}(\rho_{12}+1)-2\eta_{22}\xi_{12}(\rho_{12}+1))\sigma_2-(\eta_{12}+(1-2\eta_{22})\xi_{12}+1)\sigma_1)\delta_2+\delta_1(\delta_2(\sigma_1-(\rho_{12}+1)\sigma_2)+\sigma_2((\eta_{22}(\rho_{12}+1)-\rho_{12})\sigma_2-(\eta_{22}-1)\sigma_1))+\sigma_2^2(\eta_{12}((\eta_{22}-1)\sigma_1-(\eta_{22}(\rho_{12}+1)-\rho_{12})\sigma_2)+\eta_{22}((1-(\eta_{22}-1)\xi_{12})\sigma_1+(\eta_{22}\xi_{12}(\rho_{12}+1)-(\xi_{12}+1)\rho_{12})\sigma_2)))(\eta_{12}\eta_{22}\sigma_2^3-\eta_{12}\rho_{12}\sigma_2^3+\eta_{12}\eta_{22}\rho_{12}\sigma_2^3+\eta_{22}\rho_{12}\sigma_2^3-\delta_2\eta_{12}\sigma_2^2-\delta_1\eta_{22}\sigma_2^2+\delta_1\rho_{12}\sigma_2^2-\delta_2\rho_{12}\sigma_2^2-\delta_2\eta_{12}\rho_{12}\sigma_2^2-\delta_1\eta_{22}\rho_{12}\sigma_2^2+\eta_{12}\sigma_1\sigma_2^2-\eta_{12}\eta_{22}\sigma_1\sigma_2^2-\eta_{22}\sigma_1\sigma_2^2+\eta_{22}\xi_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2^2+\eta_{22}\xi_{12}\rho_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2^2+\xi_{12}\rho_{12}(\eta_{22}\sigma_2-\delta_2)\sigma_2^2+\delta_1\delta_2\sigma_2+\delta_1\delta_2\rho_{12}\sigma_2-\delta_1\sigma_1\sigma_2+\delta_2\sigma_1\sigma_2+\delta_2\eta_{12}\sigma_1\sigma_2+\delta_1\eta_{22}\sigma_1\sigma_2-\delta_2\xi_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2-\delta_2\xi_{12}\rho_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2+\xi_{12}\sigma_1(\delta_2-\eta_{22}\sigma_2)\sigma_2+\eta_{22}\xi_{12}\sigma_1(\eta_{22}\sigma_2-\delta_2)\sigma_2-\delta_1\delta_2\sigma_1+\delta_2\xi_{12}\sigma_1(\delta_2-\eta_{22}\sigma_2)+(-\delta_1+\delta_2-\sigma_1+\eta_{12}\sigma_2-\eta_{22}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{12}(\delta_2-\eta_{22}\sigma_2))(\xi_{12}\delta_2^2+((\eta_{12}+\rho_{12}+\xi_{12}(-2\eta_{22}+\rho_{12}+1)+1)\sigma_2-(\xi_{12}+1)\sigma_1)\delta_2-\delta_1(\delta_2-\sigma_1+(-\eta_{22}+\rho_{12}+1)\sigma_2)-\sigma_2((\eta_{12}-\eta_{22}(\xi_{12}+1)+1)\sigma_1+(-\xi_{12}\eta_{22}^2+(\xi_{12}+1)(\rho_{12}+1)\eta_{22}+\eta_{12}(\eta_{22}-\rho_{12}-1)-\rho_{12})\sigma_2)))-\sigma_2(\delta_2-\eta_{22}\sigma_2)(\delta_1-\delta_2\xi_{12}+(\eta_{22}\xi_{12}-\eta_{12})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)(-\delta_1+\delta_2-\sigma_1+\eta_{12}\sigma_2-\eta_{22}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{12}(\delta_2-\eta_{22}\sigma_2))^2$
	Δ_4	$\sigma_2(\delta_2-\eta_{22}\sigma_2)(\delta_1-\delta_2\xi_{12}+(\eta_{22}\xi_{12}-\eta_{12})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)((\xi_{12}((\rho_{12}+1)\sigma_2-\sigma_1)\delta_2^2+\sigma_2(((\xi_{12}+1)\rho_{12}+\eta_{12}(\rho_{12}+1)-2\eta_{22}\xi_{12}(\rho_{12}+1))\sigma_2-(\eta_{12}+(1-2\eta_{22})\xi_{12}+1)\sigma_1)\delta_2+\delta_1(\delta_2(\sigma_1-(\rho_{12}+1)\sigma_2)+\sigma_2((\eta_{22}(\rho_{12}+1)-\rho_{12})\sigma_2-(\eta_{22}-1)\sigma_1))+\sigma_2^2(\eta_{12}((\eta_{22}-1)\sigma_1-(\eta_{22}(\rho_{12}+1)-\rho_{12})\sigma_2)+\eta_{22}((1-(\eta_{22}-1)\xi_{12})\sigma_1+(\eta_{22}\xi_{12}(\rho_{12}+1)-(\xi_{12}+1)\rho_{12})\sigma_2)))(\eta_{12}\eta_{22}\sigma_2^3-\eta_{12}\rho_{12}\sigma_2^3+\eta_{12}\eta_{22}\rho_{12}\sigma_2^3+\eta_{22}\rho_{12}\sigma_2^3-\delta_2\eta_{12}\sigma_2^2-\delta_1\eta_{22}\sigma_2^2+\delta_1\rho_{12}\sigma_2^2-\delta_2\rho_{12}\sigma_2^2-\delta_2\eta_{12}\rho_{12}\sigma_2^2-\delta_1\eta_{22}\rho_{12}\sigma_2^2+\eta_{12}\sigma_1\sigma_2^2-\eta_{12}\eta_{22}\sigma_1\sigma_2^2-\eta_{22}\sigma_1\sigma_2^2+\eta_{22}\xi_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2^2+\eta_{22}\xi_{12}\rho_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2^2+\xi_{12}\rho_{12}(\eta_{22}\sigma_2-\delta_2)\sigma_2^2+\delta_1\delta_2\sigma_2+\delta_1\delta_2\rho_{12}\sigma_2-\delta_1\sigma_1\sigma_2+\delta_2\sigma_1\sigma_2+\delta_2\eta_{12}\sigma_1\sigma_2+\delta_1\eta_{22}\sigma_1\sigma_2-\delta_2\xi_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2-\delta_2\xi_{12}\rho_{12}(\delta_2-\eta_{22}\sigma_2)\sigma_2+\xi_{12}\sigma_1(\delta_2-\eta_{22}\sigma_2)\sigma_2+\eta_{22}\xi_{12}\sigma_1(\eta_{22}\sigma_2-\delta_2)\sigma_2-\delta_1\delta_2\sigma_1+\delta_2\xi_{12}\sigma_1(\delta_2-\eta_{22}\sigma_2)+(-\delta_1+\delta_2-\sigma_1+\eta_{12}\sigma_2-\eta_{22}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{12}(\delta_2-\eta_{22}\sigma_2))(\xi_{12}\delta_2^2+((\eta_{12}+\rho_{12}+\xi_{12}(-2\eta_{22}+\rho_{12}+1)+1)\sigma_2-(\xi_{12}+1)\sigma_1)\delta_2-\delta_1(\delta_2-\sigma_1+(-\eta_{22}+\rho_{12}+1)\sigma_2)-\sigma_2((\eta_{12}-\eta_{22}(\xi_{12}+1)+1)\sigma_1+(-\xi_{12}\eta_{22}^2+(\xi_{12}+1)(\rho_{12}+1)\eta_{22}+\eta_{12}(\eta_{22}-\rho_{12}-1)-\rho_{12})\sigma_2)))-\sigma_2(\delta_2-\eta_{22}\sigma_2)(\delta_1-\delta_2\xi_{12}+(\eta_{22}\xi_{12}-\eta_{12})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)(-\delta_1+\delta_2-\sigma_1+\eta_{12}\sigma_2-\eta_{22}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{12}(\delta_2-\eta_{22}\sigma_2))^2)$
$\dot{O}_3$	Δ_1	$\xi_{23}(\delta_3-\eta_{32}\sigma_2)-\delta_2+\delta_3+\eta_{22}\sigma_2-\eta_{32}\sigma_2+\rho_{12}\sigma_2-\sigma_1+\sigma_2$
	Δ_2	$\delta_3^2(\sigma_2(\eta_{22}(2\xi_{23}+1)-(\xi_{23}+1)(\xi_{23}(3\eta_{32}-\rho_{12}-1)-\rho_{12}-1))-(\xi_{23}+1)^2\sigma_1)+\delta_3(-2(\xi_{23}+1)\sigma_2\sigma_1(-\eta_{32}(\xi_{23}+1)+\eta_{22}+\rho_{12}+1)+\sigma_2^2(2\eta_{22}((\xi_{23}+1)(\rho_{12}+1)-\eta_{32}(2\xi_{23}+1))+(\xi_{23}+1)(-2\eta_{32}(\xi_{23}+1)(\rho_{12}+1)+3\eta_{32}^2\xi_{23}+(\rho_{12}+1)^2)+\eta_{22}^2)+(\xi_{23}+1)\sigma_1^2)-\delta_2(\delta_3+\sigma_2(-\eta_{32}+\rho_{12}+1)-\sigma_1)(\delta_3(2\xi_{23}+1)+\sigma_2(-\eta_{32}(2\xi_{23}+1)+2\eta_{22}+\rho_{12}+1)-\sigma_1)+\delta_2^2(\delta_3+\sigma_2(-\eta_{32}+\rho_{12}+1)-\sigma_1)+\delta_3^3\xi_{23}(\xi_{23}+1)+\sigma_2(-\sigma_2\sigma_1(-\eta_{32}(\xi_{23}+1)+\eta_{22}+1)(-\eta_{32}(\xi_{23}+1)+\eta_{22}+2\rho_{12}+1)+\sigma_2^2(\eta_{32}^2(\xi_{23}+1)^2(\rho_{12}+1)-\eta_{32}(\xi_{23}+1)(\rho_{12}+1)^2+\eta_{22}(\eta_{32}-\rho_{12}-1)(\eta_{32}(2\xi_{23}+1)-\rho_{12}-1)+\eta_{32}^3(-\xi_{23})(\xi_{23}+1)+\eta_{22}^2(-\eta_{32}+\rho_{12}+1)+\rho_{12}(\rho_{12}+1))+\sigma_1^2(-\eta_{32}(\xi_{23}+1)+\eta_{22}+1))$
	Δ_3	$(\xi_{23}((\rho_{12}+1)\sigma_2-\sigma_1)\delta_3^2+\sigma_2(((\xi_{23}+1)\rho_{12}+\eta_{22}(\rho_{12}+1)-2\eta_{32}\xi_{23}(\rho_{12}+1))\sigma_2-(\eta_{22}+(1-2\eta_{32})\xi_{23}+1)\sigma_1)\delta_3+\delta_2(\delta_3(\sigma_1-(\rho_{12}+1)\sigma_2)+\sigma_2((\eta_{32}(\rho_{12}+1)-\rho_{12})\sigma_2-(\eta_{32}-1)\sigma_1))+\sigma_2^2(\eta_{22}((\eta_{32}-1)\sigma_1-(\eta_{32}(\rho_{12}+1)-\rho_{12})\sigma_2)+\eta_{32}((1-(\eta_{32}-1)\xi_{23})\sigma_1+(\eta_{32}\xi_{23}(\rho_{12}+1)-(\xi_{23}+1)\rho_{12})\sigma_2)))(\eta_{22}\eta_{32}\sigma_2^3-\eta_{22}\rho_{12}\sigma_2^3+\eta_{22}\eta_{32}\rho_{12}\sigma_2^3+\eta_{32}\rho_{12}\sigma_2^3-\delta_3\eta_{22}\sigma_2^2-\delta_2\eta_{32}\sigma_2^2+\delta_2\rho_{12}\sigma_2^2-\delta_3\rho_{12}\sigma_2^2-\delta_3\eta_{22}\rho_{12}\sigma_2^2-\delta_2\eta_{32}\rho_{12}\sigma_2^2+\eta_{22}\sigma_1\sigma_2^2-\eta_{22}\eta_{32}\sigma_1\sigma_2^2-\eta_{32}\sigma_1\sigma_2^2+\eta_{32}\xi_{23}(\delta_3-\eta_{32}\sigma_2)\sigma_2^2+\eta_{32}\xi_{23}\rho_{12}(\delta_3-\eta_{32}\sigma_2)\sigma_2^2+\xi_{23}\rho_{12}(\eta_{32}\sigma_2-\delta_3)\sigma_2^2+\delta_2\delta_3\sigma_2+\delta_2\delta_3\rho_{12}\sigma_2-\delta_2\sigma_1\sigma_2+\delta_3\sigma_1\sigma_2+\delta_3\eta_{22}\sigma_1\sigma_2+\delta_2\eta_{32}\sigma_1\sigma_2-\delta_3\xi_{23}(\delta_3-\eta_{32}\sigma_2)\sigma_2-\delta_3\xi_{23}\rho_{12}(\delta_3-\eta_{32}\sigma_2)\sigma_2+\xi_{23}\sigma_1(\delta_3-\eta_{32}\sigma_2)\sigma_2+\eta_{32}\xi_{23}\sigma_1(\eta_{32}\sigma_2-\delta_3)\sigma_2-\delta_2\delta_3\sigma_1+\delta_3\xi_{23}\sigma_1(\delta_3-\eta_{32}\sigma_2)+(-\delta_2+\delta_3-\sigma_1+\eta_{22}\sigma_2-\eta_{32}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{23}(\delta_3-\eta_{32}\sigma_2))(\xi_{23}\delta_3^2+((\eta_{22}+\rho_{12}+\xi_{23}(-2\eta_{32}+\rho_{12}+1)+1)\sigma_2-(\xi_{23}+1)\sigma_1)\delta_3-\delta_2(\delta_3-\sigma_1+(-\eta_{32}+\rho_{12}+1)\sigma_2)-\sigma_2((\eta_{22}-\eta_{32}(\xi_{23}+1)+1)\sigma_1+(-\xi_{23}\eta_{32}^2+(\xi_{23}+1)(\rho_{12}+1)\eta_{32}+\eta_{22}(\eta_{32}-\rho_{12}-1)-\rho_{12})\sigma_2)))-\sigma_2(\delta_3-\eta_{32}\sigma_2)(\delta_2-\delta_3\xi_{23}+(\eta_{32}\xi_{23}-\eta_{22})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)(-\delta_2+\delta_3-\sigma_1+\eta_{22}\sigma_2-\eta_{32}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{23}(\delta_3-\eta_{32}\sigma_2))^2$
	Δ_4	$\sigma_2(\delta_3-\eta_{32}\sigma_2)(\delta_2-\delta_3\xi_{23}+(\eta_{32}\xi_{23}-\eta_{22})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)((\xi_{23}((\rho_{12}+1)\sigma_2-\sigma_1)\delta_3^2+\sigma_2(((\xi_{23}+1)\rho_{12}+\eta_{22}(\rho_{12}+1)-2\eta_{32}\xi_{23}(\rho_{12}+1))\sigma_2-(\eta_{22}+(1-2\eta_{32})\xi_{23}+1)\sigma_1)\delta_3+\delta_2(\delta_3(\sigma_1-(\rho_{12}+1)\sigma_2)+\sigma_2((\eta_{32}(\rho_{12}+1)-\rho_{12})\sigma_2-(\eta_{32}-1)\sigma_1))+\sigma_2^2(\eta_{22}((\eta_{32}-1)\sigma_1-(\eta_{32}(\rho_{12}+1)-\rho_{12})\sigma_2)+\eta_{32}((1-(\eta_{32}-1)\xi_{23})\sigma_1+(\eta_{32}\xi_{23}(\rho_{12}+1)-(\xi_{23}+1)\rho_{12})\sigma_2)))(\eta_{22}\eta_{32}\sigma_2^3-\eta_{22}\rho_{12}\sigma_2^3+\eta_{22}\eta_{32}\rho_{12}\sigma_2^3+\eta_{32}\rho_{12}\sigma_2^3-\delta_3\eta_{22}\sigma_2^2-\delta_2\eta_{32}\sigma_2^2+\delta_2\rho_{12}\sigma_2^2-\delta_3\rho_{12}\sigma_2^2-\delta_3\eta_{22}\rho_{12}\sigma_2^2-\delta_2\eta_{32}\rho_{12}\sigma_2^2+\eta_{22}\sigma_1\sigma_2^2-\eta_{22}\eta_{32}\sigma_1\sigma_2^2-\eta_{32}\sigma_1\sigma_2^2+\eta_{32}\xi_{23}(\delta_3-\eta_{32}\sigma_2)\sigma_2^2+\eta_{32}\xi_{23}\rho_{12}(\delta_3-\eta_{32}\sigma_2)\sigma_2^2+\xi_{23}\rho_{12}(\eta_{32}\sigma_2-\delta_3)\sigma_2^2+\delta_2\delta_3\sigma_2+\delta_2\delta_3\rho_{12}\sigma_2-\delta_2\sigma_1\sigma_2+\delta_3\sigma_1\sigma_2+\delta_3\eta_{22}\sigma_1\sigma_2+\delta_2\eta_{32}\sigma_1\sigma_2-\delta_3\xi_{23}(\delta_3-\eta_{32}\sigma_2)\sigma_2-\delta_3\xi_{23}\rho_{12}(\delta_3-\eta_{32}\sigma_2)\sigma_2+\xi_{23}\sigma_1(\delta_3-\eta_{32}\sigma_2)\sigma_2+\eta_{32}\xi_{23}\sigma_1(\eta_{32}\sigma_2-\delta_3)\sigma_2-\delta_2\delta_3\sigma_1+\delta_3\xi_{23}\sigma_1(\delta_3-\eta_{32}\sigma_2)+(-\delta_2+\delta_3-\sigma_1+\eta_{22}\sigma_2-\eta_{32}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{23}(\delta_3-\eta_{32}\sigma_2))(\xi_{23}\delta_3^2+((\eta_{22}+\rho_{12}+\xi_{23}(-2\eta_{32}+\rho_{12}+1)+1)\sigma_2-(\xi_{23}+1)\sigma_1)\delta_3-\delta_2(\delta_3-\sigma_1+(-\eta_{32}+\rho_{12}+1)\sigma_2)-\sigma_2((\eta_{22}-\eta_{32}(\xi_{23}+1)+1)\sigma_1+(-\xi_{23}\eta_{32}^2+(\xi_{23}+1)(\rho_{12}+1)\eta_{32}+\eta_{22}(\eta_{32}-\rho_{12}-1)-\rho_{12})\sigma_2)))-\sigma_2(\delta_3-\eta_{32}\sigma_2)(\delta_2-\delta_3\xi_{23}+(\eta_{32}\xi_{23}-\eta_{22})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)(-\delta_2+\delta_3-\sigma_1+\eta_{22}\sigma_2-\eta_{32}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{23}(\delta_3-\eta_{32}\sigma_2))^2)$

Table 3.3: The table shows the values of Δ_i for equilibrium points $\dot{O}_2, \dot{O}_3$ on 4-dimensional subspaces $x_3 = y_3 = 0$ and $x_3 = y_1 = 0$ respectively.

S	Δ_i	**Values of Δ_i for the equilibrium points $\dot{O}_1$ and $\hat{O}_2$**
$\dot{O}_1$	Δ_1	$\xi_{31}(\delta_1-\eta_{12}\sigma_2)+\delta_1-\delta_3-\eta_{12}\sigma_2+\eta_{32}\sigma_2+\rho_{12}\sigma_2-\sigma_1+\sigma_2$
	Δ_2	$\delta_1^2(-\delta_3(2\xi_{31}+1)+\sigma_2(\eta_{32}(2\xi_{31}+1)-(\xi_{31}+1)(\xi_{31}(3\eta_{12}-\rho_{12}-1)-\rho_{12}-1))+(\xi_{31}+1)^2(-\sigma_1))+\delta_1(2\delta_3((\xi_{31}+1)\sigma_1-\sigma_2(-\eta_{12}(2\xi_{31}+1)+\eta_{32}+(\xi_{31}+1)(\rho_{12}+1)))+\delta_3^2+\sigma_2^2(-2\eta_{12}(\eta_{32}(2\xi_{31}+1)+(\xi_{31}+1)^2(\rho_{12}+1))+2\eta_{32}(\xi_{31}+1)(\rho_{12}+1)+3\eta_{12}^2\xi_{31}(\xi_{31}+1)+\eta_{32}^2+(\xi_{31}+1)(\rho_{12}+1)^2)+2(\xi_{31}+1)\sigma_1\sigma_2(\eta_{12}(\xi_{31}+1)-\eta_{32}-\rho_{12}-1)+(\xi_{31}+1)\sigma_1^2)-\delta_3(\sigma_2(\eta_{12}-\rho_{12}-1)+\sigma_1)(\sigma_2(\eta_{12}(2\xi_{31}+1)-2\eta_{32}-\rho_{12}-1)+\sigma_1)+\delta_3^2(\sigma_2(-\eta_{12}+\rho_{12}+1)-\sigma_1)+\delta_1^3\xi_{31}(\xi_{31}+1)+\sigma_2(-\sigma_2\sigma_1(\eta_{12}(\xi_{31}+1)-\eta_{32}-1)(\eta_{12}(\xi_{31}+1)-\eta_{32}-2\rho_{12}-1)+\sigma_2^2(\eta_{12}^2(\eta_{32}(2\xi_{31}+1)+(\xi_{31}+1)^2(\rho_{12}+1))-\eta_{12}(2\eta_{32}(\xi_{31}+1)(\rho_{12}+1)+\eta_{32}^2+(\xi_{31}+1)(\rho_{12}+1)^2)+\eta_{12}^3(-\xi_{31})(\xi_{31}+1)+(\eta_{32}+1)(\rho_{12}+1)(\eta_{32}+\rho_{12}))+\sigma_1^2(-\eta_{12}(\xi_{31}+1)+\eta_{32}+1))$
	Δ_3	$\sigma_2(\delta_1-\eta_{12}\sigma_2)(-\delta_3+\delta_1\xi_{31}+(\eta_{32}-\eta_{12}\xi_{31})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)(\delta_3-\delta_1(\xi_{31}+1)+\sigma_1+\eta_{12}\sigma_2-\eta_{32}\sigma_2+\eta_{12}\xi_{31}\sigma_2-\rho_{12}\sigma_2-\sigma_2)^2+(\eta_{12}\eta_{32}\sigma_2^3+\eta_{12}\rho_{12}\sigma_2^3+\eta_{12}\eta_{32}\rho_{12}\sigma_2^3-\eta_{32}\rho_{12}\sigma_2^3-\delta_3\eta_{12}\sigma_2^2-\delta_1\eta_{32}\sigma_2^2-\delta_1\rho_{12}\sigma_2^2+\delta_3\rho_{12}\sigma_2^2-\delta_3\eta_{12}\rho_{12}\sigma_2^2-\delta_1\eta_{32}\rho_{12}\sigma_2^2-\eta_{12}\sigma_1\sigma_2^2-\eta_{12}\eta_{32}\sigma_1\sigma_2^2+\eta_{32}\sigma_1\sigma_2^2+\eta_{12}\xi_{31}(\delta_1-\eta_{12}\sigma_2)\sigma_2^2+\eta_{12}\xi_{31}\rho_{12}(\delta_1-\eta_{12}\sigma_2)\sigma_2^2+\xi_{31}\rho_{12}(\eta_{12}\sigma_2-\delta_1)\sigma_2^2+\delta_1\delta_3\sigma_2+\delta_1\delta_3\rho_{12}\sigma_2+\delta_1\sigma_1\sigma_2-\delta_3\sigma_1\sigma_2+\delta_3\eta_{12}\sigma_1\sigma_2+\delta_1\eta_{32}\sigma_1\sigma_2-\delta_1\xi_{31}(\delta_1-\eta_{12}\sigma_2)\sigma_2-\delta_1\xi_{31}\rho_{12}(\delta_1-\eta_{12}\sigma_2)\sigma_2+\xi_{31}\sigma_1(\delta_1-\eta_{12}\sigma_2)\sigma_2+\eta_{12}\xi_{31}\sigma_1(\eta_{12}\sigma_2-\delta_1)\sigma_2-\delta_1\delta_3\sigma_1+\delta_1\xi_{31}\sigma_1(\delta_1-\eta_{12}\sigma_2)+(\delta_1-\delta_3-\sigma_1-\eta_{12}\sigma_2+\eta_{32}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{31}(\delta_1-\eta_{12}\sigma_2))(\xi_{31}\delta_1^2+(-\delta_3-(\xi_{31}+1)\sigma_1+(\eta_{32}+\rho_{12}+\xi_{31}(-2\eta_{12}+\rho_{12}+1)+1)\sigma_2)\delta_1+\delta_3(\sigma_1+(\eta_{12}-\rho_{12}-1)\sigma_2)+\sigma_2((-\eta_{32}+\eta_{12}(\xi_{31}+1)-1)\sigma_1+(\xi_{31}\eta_{12}^2-(\eta_{32}+(\xi_{31}+1)(\rho_{12}+1))\eta_{12}+\rho_{12}+\eta_{32}(\rho_{12}+1))\sigma_2)))(\xi_{31}((\rho_{12}+1)\sigma_2-\sigma_1)\delta_1^2+(\delta_3(\sigma_1-(\rho_{12}+1)\sigma_2)+\sigma_2(((\xi_{31}+1)\rho_{12}+\eta_{32}(\rho_{12}+1)-2\eta_{12}\xi_{31}(\rho_{12}+1))\sigma_2-(\eta_{32}+(1-2\eta_{12})\xi_{31}+1)\sigma_1))\delta_1+\sigma_2(\delta_3((\eta_{12}(\rho_{12}+1)-\rho_{12})\sigma_2-(\eta_{12}-1)\sigma_1)+\sigma_2(\xi_{31}((\rho_{12}+1)\sigma_2-\sigma_1)\eta_{12}^2+((\eta_{32}+\xi_{31}+1)\sigma_1-((\xi_{31}+1)\rho_{12}+\eta_{32}(\rho_{12}+1))\sigma_2)\eta_{12}+\eta_{32}(\rho_{12}\sigma_2-\sigma_1))))$
	Δ_4	$\sigma_2(\delta_1-\eta_{12}\sigma_2)(-\delta_3+\delta_1\xi_{31}+(\eta_{32}-\eta_{12}\xi_{31})\sigma_2)(\rho_{12}\sigma_2-\sigma_1)(\sigma_2(\delta_1-\eta_{12}\sigma_2)(-\delta_3+\delta_1\xi_{31}+(\eta_{32}-\eta_{12}\xi_{31})\sigma_2)(\sigma_1-\rho_{12}\sigma_2)(\delta_3-\delta_1(\xi_{31}+1)+\sigma_1+\eta_{12}\sigma_2-\eta_{32}\sigma_2+\eta_{12}\xi_{31}\sigma_2-\rho_{12}\sigma_2-\sigma_2)^2+(\eta_{12}\eta_{32}\sigma_2^3+\eta_{12}\rho_{12}\sigma_2^3+\eta_{12}\eta_{32}\rho_{12}\sigma_2^3-\eta_{32}\rho_{12}\sigma_2^3-\delta_3\eta_{12}\sigma_2^2-\delta_1\eta_{32}\sigma_2^2-\delta_1\rho_{12}\sigma_2^2+\delta_3\rho_{12}\sigma_2^2-\delta_3\eta_{12}\rho_{12}\sigma_2^2-\delta_1\eta_{32}\rho_{12}\sigma_2^2-\eta_{12}\sigma_1\sigma_2^2-\eta_{12}\eta_{32}\sigma_1\sigma_2^2+\eta_{32}\sigma_1\sigma_2^2+\eta_{12}\xi_{31}(\delta_1-\eta_{12}\sigma_2)\sigma_2^2+\eta_{12}\xi_{31}\rho_{12}(\delta_1-\eta_{12}\sigma_2)\sigma_2^2+\xi_{31}\rho_{12}(\eta_{12}\sigma_2-\delta_1)\sigma_2^2+\delta_1\delta_3\sigma_2+\delta_1\delta_3\rho_{12}\sigma_2+\delta_1\sigma_1\sigma_2-\delta_3\sigma_1\sigma_2+\delta_3\eta_{12}\sigma_1\sigma_2+\delta_1\eta_{32}\sigma_1\sigma_2-\delta_1\xi_{31}(\delta_1-\eta_{12}\sigma_2)\sigma_2-\delta_1\xi_{31}\rho_{12}(\delta_1-\eta_{12}\sigma_2)\sigma_2+\xi_{31}\sigma_1(\delta_1-\eta_{12}\sigma_2)\sigma_2+\eta_{12}\xi_{31}\sigma_1(\eta_{12}\sigma_2-\delta_1)\sigma_2-\delta_1\delta_3\sigma_1+\delta_1\xi_{31}\sigma_1(\delta_1-\eta_{12}\sigma_2)+(\delta_1-\delta_3-\sigma_1-\eta_{12}\sigma_2+\eta_{32}\sigma_2+\rho_{12}\sigma_2+\sigma_2+\xi_{31}(\delta_1-\eta_{12}\sigma_2))(\xi_{31}\delta_1^2+(-\delta_3-(\xi_{31}+1)\sigma_1+(\eta_{32}+\rho_{12}+\xi_{31}(-2\eta_{12}+\rho_{12}+1)+1)\sigma_2)\delta_1+\delta_3(\sigma_1+(\eta_{12}-\rho_{12}-1)\sigma_2)+\sigma_2((-\eta_{32}+\eta_{12}(\xi_{31}+1)-1)\sigma_1+(\xi_{31}\eta_{12}^2-(\eta_{32}+(\xi_{31}+1)(\rho_{12}+1))\eta_{12}+\rho_{12}+\eta_{32}(\rho_{12}+1))\sigma_2)))(\xi_{31}((\rho_{12}+1)\sigma_2-\sigma_1)\delta_1^2+(\delta_3(\sigma_1-(\rho_{12}+1)\sigma_2)+\sigma_2(((\xi_{31}+1)\rho_{12}+\eta_{32}(\rho_{12}+1)-2\eta_{12}\xi_{31}(\rho_{12}+1))\sigma_2-(\eta_{32}+(1-2\eta_{12})\xi_{31}+1)\sigma_1))\delta_1+\sigma_2(\delta_3((\eta_{12}(\rho_{12}+1)-\rho_{12})\sigma_2-(\eta_{12}-1)\sigma_1)+\sigma_2(\xi_{31}((\rho_{12}+1)\sigma_2-\sigma_1)\eta_{12}^2+((\eta_{32}+\xi_{31}+1)\sigma_1-((\xi_{31}+1)\rho_{12}+\eta_{32}(\rho_{12}+1))\sigma_2)\eta_{12}+\eta_{32}(\rho_{12}\sigma_2-\sigma_1)))))$
$\hat{O}_2$	Δ_1	$\xi_{12}(\delta_2-\eta_{23}\sigma_3)-\delta_1+\delta_2+\eta_{13}\sigma_3-\eta_{23}\sigma_3+\rho_{23}\sigma_3-\sigma_2+\sigma_3$
	Δ_2	$\delta_2^2(\sigma_3(\eta_{13}(2\xi_{12}+1)-(\xi_{12}+1)(\xi_{12}(3\eta_{23}-\rho_{23}-1)-\rho_{23}-1))-(\xi_{12}+1)^2\sigma_2)+\delta_2(-2(\xi_{12}+1)\sigma_3\sigma_2(-\eta_{23}(\xi_{12}+1)+\eta_{13}+\rho_{23}+1)+\sigma_3^2(2\eta_{13}((\xi_{12}+1)(\rho_{23}+1)-\eta_{23}(2\xi_{12}+1))+(\xi_{12}+1)(-2\eta_{23}(\xi_{12}+1)(\rho_{23}+1)+3\eta_{23}^2\xi_{12}+(\rho_{23}+1)^2)+\eta_{13}^2)+(\xi_{12}+1)\sigma_2^2)-\delta_1(\delta_2+\sigma_3(-\eta_{23}+\rho_{23}+1)-\sigma_2)(\delta_2(2\xi_{12}+1)+\sigma_3(-\eta_{23}(2\xi_{12}+1)+2\eta_{13}+\rho_{23}+1)-\sigma_2)+\delta_1^2(\delta_2+\sigma_3(-\eta_{23}+\rho_{23}+1)-\sigma_2)+\delta_2^3\xi_{12}(\xi_{12}+1)+\sigma_3(-\sigma_3\sigma_2(-\eta_{23}(\xi_{12}+1)+\eta_{13}+1)(-\eta_{23}(\xi_{12}+1)+\eta_{13}+2\rho_{23}+1)+\sigma_3^2(\eta_{23}^2(\xi_{12}+1)^2(\rho_{23}+1)-\eta_{23}(\xi_{12}+1)(\rho_{23}+1)^2+\eta_{13}(\eta_{23}-\rho_{23}-1)(\eta_{23}(2\xi_{12}+1)-\rho_{23}-1)+\eta_{23}^3(-\xi_{12})(\xi_{12}+1)+\eta_{13}^2(-\eta_{23}+\rho_{23}+1)+\rho_{23}(\rho_{23}+1))+\sigma_2^2(-\eta_{23}(\xi_{12}+1)+\eta_{13}+1))$
	Δ_3	$(\xi_{12}((\rho_{23}+1)\sigma_3-\sigma_2)\delta_2^2+\sigma_3(((\xi_{12}+1)\rho_{23}+\eta_{13}(\rho_{23}+1)-2\eta_{23}\xi_{12}(\rho_{23}+1))\sigma_3-(\eta_{13}+(1-2\eta_{23})\xi_{12}+1)\sigma_2)\delta_2+\delta_1(\delta_2(\sigma_2-(\rho_{23}+1)\sigma_3)+\sigma_3((\eta_{23}(\rho_{23}+1)-\rho_{23})\sigma_3-(\eta_{23}-1)\sigma_2))+\sigma_3^2(\eta_{13}((\eta_{23}-1)\sigma_2-(\eta_{23}(\rho_{23}+1)-\rho_{23})\sigma_3)+\eta_{23}((1-(\eta_{23}-1)\xi_{12})\sigma_2+(\eta_{23}\xi_{12}(\rho_{23}+1)-(\xi_{12}+1)\rho_{23})\sigma_3)))(\eta_{13}\eta_{23}\sigma_3^3-\eta_{13}\rho_{23}\sigma_3^3+\eta_{13}\eta_{23}\rho_{23}\sigma_3^3+\eta_{23}\rho_{23}\sigma_3^3-\delta_2\eta_{13}\sigma_3^2-\delta_1\eta_{23}\sigma_3^2+\delta_1\rho_{23}\sigma_3^2-\delta_2\rho_{23}\sigma_3^2-\delta_2\eta_{13}\rho_{23}\sigma_3^2-\delta_1\eta_{23}\rho_{23}\sigma_3^2+\eta_{13}\sigma_2\sigma_3^2-\eta_{13}\eta_{23}\sigma_2\sigma_3^2-\eta_{23}\sigma_2\sigma_3^2+\eta_{23}\xi_{12}(\delta_2-\eta_{23}\sigma_3)\sigma_3^2+\eta_{23}\xi_{12}\rho_{23}(\delta_2-\eta_{23}\sigma_3)\sigma_3^2+\xi_{12}\rho_{23}(\eta_{23}\sigma_3-\delta_2)\sigma_3^2+\delta_1\delta_2\sigma_3+\delta_1\delta_2\rho_{23}\sigma_3-\delta_1\sigma_2\sigma_3+\delta_2\sigma_2\sigma_3+\delta_2\eta_{13}\sigma_2\sigma_3+\delta_1\eta_{23}\sigma_2\sigma_3-\delta_2\xi_{12}(\delta_2-\eta_{23}\sigma_3)\sigma_3-\delta_2\xi_{12}\rho_{23}(\delta_2-\eta_{23}\sigma_3)\sigma_3+\xi_{12}\sigma_2(\delta_2-\eta_{23}\sigma_3)\sigma_3+\eta_{23}\xi_{12}\sigma_2(\eta_{23}\sigma_3-\delta_2)\sigma_3-\delta_1\delta_2\sigma_2+\delta_2\xi_{12}\sigma_2(\delta_2-\eta_{23}\sigma_3)+(-\delta_1+\delta_2-\sigma_2+\eta_{13}\sigma_3-\eta_{23}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{12}(\delta_2-\eta_{23}\sigma_3))(\xi_{12}\delta_2^2+((\eta_{13}+\rho_{23}+\xi_{12}(-2\eta_{23}+\rho_{23}+1)+1)\sigma_3-(\xi_{12}+1)\sigma_2)\delta_2-\delta_1(\delta_2-\sigma_2+(-\eta_{23}+\rho_{23}+1)\sigma_3)-\sigma_3((\eta_{13}-\eta_{23}(\xi_{12}+1)+1)\sigma_2+(-\xi_{12}\eta_{23}^2+(\xi_{12}+1)(\rho_{23}+1)\eta_{23}+\eta_{13}(\eta_{23}-\rho_{23}-1)-\rho_{23})\sigma_3)))-\sigma_3(\delta_2-\eta_{23}\sigma_3)(\delta_1-\delta_2\xi_{12}+(\eta_{23}\xi_{12}-\eta_{13})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)(-\delta_1+\delta_2-\sigma_2+\eta_{13}\sigma_3-\eta_{23}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{12}(\delta_2-\eta_{23}\sigma_3))^2$
	Δ_4	$\sigma_3(\delta_2-\eta_{23}\sigma_3)(\delta_1-\delta_2\xi_{12}+(\eta_{23}\xi_{12}-\eta_{13})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)((\xi_{12}((\rho_{23}+1)\sigma_3-\sigma_2)\delta_2^2+\sigma_3(((\xi_{12}+1)\rho_{23}+\eta_{13}(\rho_{23}+1)-2\eta_{23}\xi_{12}(\rho_{23}+1))\sigma_3-(\eta_{13}+(1-2\eta_{23})\xi_{12}+1)\sigma_2)\delta_2+\delta_1(\delta_2(\sigma_2-(\rho_{23}+1)\sigma_3)+\sigma_3((\eta_{23}(\rho_{23}+1)-\rho_{23})\sigma_3-(\eta_{23}-1)\sigma_2))+\sigma_3^2(\eta_{13}((\eta_{23}-1)\sigma_2-(\eta_{23}(\rho_{23}+1)-\rho_{23}\sigma_3)+\eta_{23}((1-(\eta_{23}-1)\xi_{12})\sigma_2+(\eta_{23}\xi_{12}(\rho_{23}+)-(\xi_{12}+1)\rho_{23})\sigma_3)))(\eta_{13}\eta_{23}\sigma_3^3-\eta_{13}\rho_{23}\sigma_3^3+\eta_{13}\eta_{23}\rho_{23}\sigma_3^3+\eta_{23}\rho_{23}\sigma_3^3-\delta_2\eta_{13}\sigma_3^2-\delta_1\eta_{23}\sigma_3^2+\delta_1\rho_{23}\sigma_3^2-\delta_2\rho_{23}\sigma_3^2-\delta_2\eta_{13}\rho_{23}\sigma_3^2-\delta_1\eta_{23}\rho_{23}\sigma_3^2+\eta_{13}\sigma_2\sigma_3^2-\eta_{13}\eta_{23}\sigma_2\sigma_3^2-\eta_{23}\sigma_2\sigma_3^2+\eta_{23}\xi_{12}(\delta_2-\eta_{23}\sigma_3)\sigma_3^2+\eta_{23}\xi_{12}\rho_{23}(\delta_2-\eta_{23}\sigma_3)\sigma_3^2+\xi_{12}\rho_{23}(\eta_{23}\sigma_3-\delta_2)\sigma_3^2+\delta_1\delta_2\sigma_3+\delta_1\delta_2\rho_{23}\sigma_3-\delta_1\sigma_2\sigma_3+\delta_2\sigma_2\sigma_3+\delta_2\eta_{13}\sigma_2\sigma_3+\delta_1\eta_{23}\sigma_2\sigma_3-\delta_2\xi_{12}(\delta_2-\eta_{23}\sigma_3)\sigma_3-\delta_2\xi_{12}\rho_{23}(\delta_2-\eta_{23}\sigma_3)\sigma_3+\xi_{12}\sigma_2(\delta_2-\eta_{23}\sigma_3)\sigma_3+\eta_{23}\xi_{12}\sigma_2(\eta_{23}\sigma_3-\delta_2)\sigma_3-\delta_1\delta_2\sigma_2+\delta_2\xi_{12}\sigma_2(\delta_2-\eta_{23}\sigma_3)+(-\delta_1+\delta_2-\sigma_2+\eta_{13}\sigma_3-\eta_{23}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{12}(\delta_2-\eta_{23}\sigma_3))(\xi_{12}\delta_2^2+((\eta_{13}+\rho_{23}+\xi_{12}(-2\eta_{23}+\rho_{23}+1)+1)\sigma_3-(\xi_{12}+1)\sigma_2)\delta_2-\delta_1(\delta_2-\sigma_2+(-\eta_{23}+\rho_{23}+1)\sigma_3)-\sigma_3((\eta_{13}-\eta_{23}(\xi_{12}+1)+1)\sigma_2+(-\xi_{12}\eta_{23}^2+(\xi_{12}+1)(\rho_{23}+1)\eta_{23}+\eta_{13}(\eta_{23}-\rho_{23}-1)-\rho_{23})\sigma_3)))-\sigma_3(\delta_2-\eta_{23}\sigma_3)(\delta_1-\delta_2\xi_{12}+(\eta_{23}\xi_{12}-\eta_{13})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)(-\delta_1+\delta_2-\sigma_2+\eta_{13}\sigma_3-\eta_{23}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{12}(\delta_2-\eta_{23}\sigma_3))^2)$

Table 3.4: The table shows the values of Δ_i for equilibrium points $\dot{O}_1, \hat{O}_2$ on 4-dimensional subspaces $x_3 = y_2 = 0$ and $x_1 = y_3 = 0$ respectively.

S	Δ_i	**Values of Δ_i for the equilibrium points $\hat{O}_3$ and $\hat{O}_1$**
$\hat{O}_3$	Δ_1	$\xi_{23}\,(\delta_3-\eta_{33}\sigma_3)-\delta_2+\delta_3+\eta_{23}\sigma_3-\eta_{33}\sigma_3+\rho_{23}\sigma_3-\sigma_2+\sigma_3$
	Δ_2	$\delta_3^2(\sigma_3(\eta_{23}(2\xi_{23}+1)-(\xi_{23}+1)(\xi_{23}(3\eta_{33}-\rho_{23}-1)-\rho_{23}-1))-(\xi_{23}+1)^2\sigma_2)+\delta_3(-2(\xi_{23}+1)\sigma_3\sigma_2(-\eta_{33}(\xi_{23}+1)+\eta_{23}+\rho_{23}+1)+\sigma_3^2(2\eta_{23}((\xi_{23}+1)(\rho_{23}+1)-\eta_{33}(2\xi_{23}+1))+(\xi_{23}+1)(-2\eta_{33}(\xi_{23}+1)(\rho_{23}+1)+3\eta_{33}^2\xi_{23}+(\rho_{23}+1)^2)+\eta_{23}^2)+(\xi_{23}+1)\sigma_2^2)-\delta_2(\delta_3+\sigma_3(-\eta_{33}+\rho_{23}+1)-\sigma_2)(\delta_3(2\xi_{23}+1)+\sigma_3(-\eta_{33}(2\xi_{23}+1)+2\eta_{23}+\rho_{23}+1)-\sigma_2)+\delta_2^2(\delta_3+\sigma_3(-\eta_{33}+\rho_{23}+1)-\sigma_2)+\delta_3^3\xi_{23}(\xi_{23}+1)+\sigma_3(-\sigma_3\sigma_2(-\eta_{33}(\xi_{23}+1)+\eta_{23}+1)(-\eta_{33}(\xi_{23}+1)+\eta_{23}+2\rho_{23}+1)+\sigma_3^2(\eta_{33}^2(\xi_{23}+1)^2(\rho_{23}+1)-\eta_{33}(\xi_{23}+1)(\rho_{23}+1)^2+\eta_{23}(\eta_{33}-\rho_{23}-1)(\eta_{33}(2\xi_{23}+1)-\rho_{23}-1)+\eta_{33}^3(-\xi_{23})(\xi_{23}+1)+\eta_{23}^2(-\eta_{33}+\rho_{23}+1)+\rho_{23}(\rho_{23}+1))+\sigma_2^2(-\eta_{33}(\xi_{23}+1)+\eta_{23}+1))$
	Δ_3	$(\xi_{23}((\rho_{23}+1)\sigma_3-\sigma_2)\delta_3^2+\sigma_3(((\xi_{23}+1)\rho_{23}+\eta_{23}(\rho_{23}+1)-2\eta_{33}\xi_{23}(\rho_{23}+1))\sigma_3-(\eta_{23}+(1-2\eta_{33})\xi_{23}+1)\sigma_2)\delta_3+\delta_2(\delta_3(\sigma_2-(\rho_{23}+1)\sigma_3)+\sigma_3((\eta_{33}(\rho_{23}+1)-\rho_{23})\sigma_3-(\eta_{33}-1)\sigma_2))+\sigma_3^2(\eta_{23}((\eta_{33}-1)\sigma_2-(\eta_{33}(\rho_{23}+1)-\rho_{23})\sigma_3)+\eta_{33}((1-(\eta_{33}-1)\xi_{23})\sigma_2+(\eta_{33}\xi_{23}(\rho_{23}+1)-(\xi_{23}+1)\rho_{23})\sigma_3)))(\eta_{23}\eta_{33}\sigma_3^3-\eta_{23}\rho_{23}\sigma_3^3+\eta_{23}\eta_{33}\rho_{23}\sigma_3^3+\eta_{33}\rho_{23}\sigma_3^3-\delta_3\eta_{23}\sigma_3^2-\delta_2\eta_{33}\sigma_3^2+\delta_2\rho_{23}\sigma_3^2-\delta_3\rho_{23}\sigma_3^2-\delta_3\eta_{23}\rho_{23}\sigma_3^2-\delta_2\eta_{33}\rho_{23}\sigma_3^2+\eta_{23}\sigma_2\sigma_3^2-\eta_{23}\eta_{33}\sigma_2\sigma_3^2-\eta_{33}\sigma_2\sigma_3^2+\eta_{33}\xi_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3^2+\eta_{33}\xi_{23}\rho_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3^2+\xi_{23}\rho_{23}(\eta_{33}\sigma_3-\delta_3)\sigma_3^2+\delta_2\delta_3\sigma_3+\delta_2\delta_3\rho_{23}\sigma_3-\delta_2\sigma_2\sigma_3+\delta_3\sigma_2\sigma_3+\delta_3\eta_{23}\sigma_2\sigma_3+\delta_2\eta_{33}\sigma_2\sigma_3-\delta_3\xi_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3-\delta_3\xi_{23}\rho_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3+\xi_{23}\sigma_2(\delta_3-\eta_{33}\sigma_3)\sigma_3+\eta_{33}\xi_{23}\sigma_2(\eta_{33}\sigma_3-\delta_3)\sigma_3-\delta_2\delta_3\sigma_2+\delta_3\xi_{23}\sigma_2(\delta_3-\eta_{33}\sigma_3)+(-\delta_2+\delta_3-\sigma_2+\eta_{23}\sigma_3-\eta_{33}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{23}(\delta_3-\eta_{33}\sigma_3))(\xi_{23}\delta_3^2+((\eta_{23}+\rho_{23}+\xi_{23}(-2\eta_{33}+\rho_{23}+1)+1)\sigma_3-(\xi_{23}+1)\sigma_2)\delta_3-\delta_2(\delta_3-\sigma_2+(-\eta_{33}+\rho_{23}+1)\sigma_3)-\sigma_3((\eta_{23}-\eta_{33}(\xi_{23}+1)+1)\sigma_2+(-\xi_{23}\eta_{33}^2+(\xi_{23}+1)(\rho_{23}+1)\eta_{33}+\eta_{23}(\eta_{33}-\rho_{23}-1)-\rho_{23})\sigma_3)))-\sigma_3(\delta_3-\eta_{33}\sigma_3)(\delta_2-\delta_3\xi_{23}+(\eta_{33}\xi_{23}-\eta_{23})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)(-\delta_2+\delta_3-\sigma_2+\eta_{23}\sigma_3-\eta_{33}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{23}(\delta_3-\eta_{33}\sigma_3))^2$
	Δ_4	$\sigma_3(\delta_3-\eta_{33}\sigma_3)(\delta_2-\delta_3\xi_{23}+(\eta_{33}\xi_{23}-\eta_{23})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)((\xi_{23}((\rho_{23}+1)\sigma_3-\sigma_2)\delta_3^2+\sigma_3(((\xi_{23}+1)\rho_{23}+\eta_{23}(\rho_{23}+1)-2\eta_{33}\xi_{23}(\rho_{23}+1))\sigma_3-(\eta_{23}+(1-2\eta_{33})\xi_{23}+1)\sigma_2)\delta_3+\delta_2(\delta_3(\sigma_2-(\rho_{23}+1)\sigma_3)+\sigma_3((\eta_{33}(\rho_{23}+1)-\rho_{23})\sigma_3-(\eta_{33}-1)\sigma_2))+\sigma_3^2(\eta_{23}((\eta_{33}-1)\sigma_2-(\eta_{33}(\rho_{23}+1)-\rho_{23})\sigma_3)+\eta_{33}((1-(\eta_{33}-1)\xi_{23})\sigma_2+(\eta_{33}\xi_{23}(\rho_{23}+1)-(\xi_{23}+1)\rho_{23})\sigma_3)))(\eta_{23}\eta_{33}\sigma_3^3-\eta_{23}\rho_{23}\sigma_3^3+\eta_{23}\eta_{33}\rho_{23}\sigma_3^3+\eta_{33}\rho_{23}\sigma_3^3-\delta_3\eta_{23}\sigma_3^2-\delta_2\eta_{33}\sigma_3^2+\delta_2\rho_{23}\sigma_3^2-\delta_3\rho_{23}\sigma_3^2-\delta_3\eta_{23}\rho_{23}\sigma_3^2-\delta_2\eta_{33}\rho_{23}\sigma_3^2+\eta_{23}\sigma_2\sigma_3^2-\eta_{23}\eta_{33}\sigma_2\sigma_3^2-\eta_{33}\sigma_2\sigma_3^2+\eta_{33}\xi_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3^2+\eta_{33}\xi_{23}\rho_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3^2+\xi_{23}\rho_{23}(\eta_{33}\sigma_3-\delta_3)\sigma_3^2+\delta_2\delta_3\sigma_3+\delta_2\delta_3\rho_{23}\sigma_3-\delta_2\sigma_2\sigma_3+\delta_3\sigma_2\sigma_3+\delta_3\eta_{23}\sigma_2\sigma_3+\delta_2\eta_{33}\sigma_2\sigma_3-\delta_3\xi_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3-\delta_3\xi_{23}\rho_{23}(\delta_3-\eta_{33}\sigma_3)\sigma_3+\xi_{23}\sigma_2(\delta_3-\eta_{33}\sigma_3)\sigma_3+\eta_{33}\xi_{23}\sigma_2(\eta_{33}\sigma_3-\delta_3)\sigma_3-\delta_2\delta_3\sigma_2+\delta_3\xi_{23}\sigma_2(\delta_3-\eta_{33}\sigma_3)+(-\delta_2+\delta_3-\sigma_2+\eta_{23}\sigma_3-\eta_{33}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{23}(\delta_3-\eta_{33}\sigma_3))(\xi_{23}\delta_3^2+((\eta_{23}+\rho_{23}+\xi_{23}(-2\eta_{33}+\rho_{23}+1)+1)\sigma_3-(\xi_{23}+1)\sigma_2)\delta_3-\delta_2(\delta_3-\sigma_2+(-\eta_{33}+\rho_{23}+1)\sigma_3)-\sigma_3((\eta_{23}-\eta_{33}(\xi_{23}+1)+1)\sigma_2+(-\xi_{23}\eta_{33}^2+(\xi_{23}+1)(\rho_{23}+1)\eta_{33}+\eta_{23}(\eta_{33}-\rho_{23}-1)-\rho_{23})\sigma_3)))-\sigma_3(\delta_3-\eta_{33}\sigma_3)(\delta_2-\delta_3\xi_{23}+(\eta_{33}\xi_{23}-\eta_{23})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)(-\delta_2+\delta_3-\sigma_2+\eta_{23}\sigma_3-\eta_{33}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{23}(\delta_3-\eta_{33}\sigma_3))^2)$
$\hat{O}_1$	Δ_1	$\xi_{31}\,(\delta_1-\eta_{13}\sigma_3)+\delta_1-\delta_3-\eta_{13}\sigma_3+\eta_{33}\sigma_3+\rho_{23}\sigma_3-\sigma_2+\sigma_3$
	Δ_2	$\delta_1^2(-\delta_3(2\xi_{31}+1)+\sigma_3(\eta_{33}(2\xi_{31}+1)-(\xi_{31}+1)(\xi_{31}(3\eta_{13}-\rho_{23}-1)-\rho_{23}-1))+(\xi_{31}+1)^2(-\sigma_2))+\delta_1(2\delta_3((\xi_{31}+1)\sigma_2-\sigma_3(-\eta_{13}(2\xi_{31}+1)+\eta_{33}+(\xi_{31}+1)(\rho_{23}+1)))+\delta_3^2+\sigma_3^2(-2\eta_{13}(\eta_{33}(2\xi_{31}+1)+(\xi_{31}+1)^2(\rho_{23}+1))+2\eta_{33}(\xi_{31}+1)(\rho_{23}+1)+3\eta_{13}^2\xi_{31}(\xi_{31}+1)+\eta_{33}^2+(\xi_{31}+1)(\rho_{23}+1)^2)+2(\xi_{31}+1)\sigma_2\sigma_3(\eta_{13}(\xi_{31}+1)-\eta_{33}-\rho_{23}-1)+(\xi_{31}+1)\sigma_2^2)-\delta_3(\sigma_3(\eta_{13}-\rho_{23}-1)+\sigma_2)(\sigma_3(\eta_{13}(2\xi_{31}+1)-2\eta_{33}-\rho_{23}-1)+\sigma_2)+\delta_3^2(\sigma_3(-\eta_{13}+\rho_{23}+1)-\sigma_2)+\delta_1^3\xi_{31}(\xi_{31}+1)+\sigma_3(-\sigma_3\sigma_2(\eta_{13}(\xi_{31}+1)-\eta_{33}-1)(\eta_{13}(\xi_{31}+1)-\eta_{33}-2\rho_{23}-1)+\sigma_3^2(\eta_{13}^2(\eta_{33}(2\xi_{31}+1)+(\xi_{31}+1)^2(\rho_{23}+1))-\eta_{13}(2\eta_{33}(\xi_{31}+1)(\rho_{23}+1)+\eta_{33}^2+(\xi_{31}+1)(\rho_{23}+1)^2)+\eta_{13}^3(-\xi_{31})(\xi_{31}+1)+(\eta_{33}+1)(\rho_{23}+1)(\eta_{33}+\rho_{23}))+\sigma_2^2(-\eta_{13}(\xi_{31}+1)+\eta_{33}+1))$
	Δ_3	$\sigma_3(\delta_1-\eta_{13}\sigma_3)(-\delta_3+\delta_1\xi_{31}+(\eta_{33}-\eta_{13}\xi_{31})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)(\delta_3-\delta_1(\xi_{31}+1)+\sigma_2+\eta_{13}\sigma_3-\eta_{33}\sigma_3+\eta_{13}\xi_{31}\sigma_3-\rho_{23}\sigma_3-\sigma_3)^2+(\eta_{13}\eta_{33}\sigma_3^3+\eta_{13}\rho_{23}\sigma_3^3+\eta_{13}\eta_{33}\rho_{23}\sigma_3^3-\eta_{33}\rho_{23}\sigma_3^3-\delta_3\eta_{13}\sigma_3^2-\delta_1\eta_{33}\sigma_3^2-\delta_1\rho_{23}\sigma_3^2+\delta_3\rho_{23}\sigma_3^2-\delta_3\eta_{13}\rho_{23}\sigma_3^2-\delta_1\eta_{33}\rho_{23}\sigma_3^2-\eta_{13}\sigma_2\sigma_3^2-\eta_{13}\eta_{33}\sigma_2\sigma_3^2+\eta_{33}\sigma_2\sigma_3^2+\eta_{13}\xi_{31}(\delta_1-\eta_{13}\sigma_3)\sigma_3^2+\eta_{13}\xi_{31}\rho_{23}(\delta_1-\eta_{13}\sigma_3)\sigma_3^2+\xi_{31}\rho_{23}(\eta_{13}\sigma_3-\delta_1)\sigma_3^2+\delta_1\delta_3\sigma_3+\delta_1\delta_3\rho_{23}\sigma_3+\delta_1\sigma_2\sigma_3-\delta_3\sigma_2\sigma_3+\delta_3\eta_{13}\sigma_2\sigma_3+\delta_1\eta_{33}\sigma_2\sigma_3-\delta_1\xi_{31}(\delta_1-\eta_{13}\sigma_3)\sigma_3-\delta_1\xi_{31}\rho_{23}(\delta_1-\eta_{13}\sigma_3)\sigma_3+\xi_{31}\sigma_2(\delta_1-\eta_{13}\sigma_3)\sigma_3+\eta_{13}\xi_{31}\sigma_2(\eta_{13}\sigma_3-\delta_1)\sigma_3-\delta_1\delta_3\sigma_2+\delta_1\xi_{31}\sigma_2(\delta_1-\eta_{13}\sigma_3)+(\delta_1-\delta_3-\sigma_2-\eta_{13}\sigma_3+\eta_{33}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{31}(\delta_1-\eta_{13}\sigma_3))(\xi_{31}\delta_1^2+(-\delta_3-(\xi_{31}+1)\sigma_2+(\eta_{33}+\rho_{23}+\xi_{31}(-2\eta_{13}+\rho_{23}+1)+1)\sigma_3)\delta_1+\delta_3(\sigma_2+(\eta_{13}-\rho_{23}-1)\sigma_3)+\sigma_3((-\eta_{33}+\eta_{13}(\xi_{31}+1)-1)\sigma_2+(\xi_{31}\eta_{13}^2-(\eta_{33}+(\xi_{31}+1)(\rho_{23}+1))\eta_{13}+\rho_{23}+\eta_{33}(\rho_{23}+1))\sigma_3)))(\xi_{31}((\rho_{23}+1)\sigma_3-\sigma_2)\delta_1^2+(\delta_3(\sigma_2-(\rho_{23}+1)\sigma_3)+\sigma_3(((\xi_{31}+1)\rho_{23}+\eta_{33}(\rho_{23}+1)-2\eta_{13}\xi_{31}(\rho_{23}+1))\sigma_3-(\eta_{33}+(1-2\eta_{13})\xi_{31}+1)\sigma_2))\delta_1+\sigma_3(\delta_3((\eta_{13}(\rho_{23}+1)-\rho_{23})\sigma_3-(\eta_{13}-1)\sigma_2)+\sigma_3(\xi_{31}((\rho_{23}+1)\sigma_3-\sigma_2)\eta_{13}^2+((\eta_{33}+\xi_{31}+1)\sigma_2-((\xi_{31}+1)\rho_{23}+\eta_{33}(\rho_{23}+1))\sigma_3)\eta_{13}+\eta_{33}(\rho_{23}\sigma_3-\sigma_2))))$
	Δ_4	$\sigma_3(\delta_1-\eta_{13}\sigma_3)(-\delta_3+\delta_1\xi_{31}+(\eta_{33}-\eta_{13}\xi_{31})\sigma_3)(\rho_{23}\sigma_3-\sigma_2)(\sigma_3(\delta_1-\eta_{13}\sigma_3)(-\delta_3+\delta_1\xi_{31}+(\eta_{33}-\eta_{13}\xi_{31})\sigma_3)(\sigma_2-\rho_{23}\sigma_3)(\delta_3-\delta_1(\xi_{31}+1)+\sigma_2+\eta_{13}\sigma_3-\eta_{33}\sigma_3+\eta_{13}\xi_{31}\sigma_3-\rho_{23}\sigma_3-\sigma_3)^2+(\eta_{13}\eta_{33}\sigma_3^3+\eta_{13}\rho_{23}\sigma_3^3+\eta_{13}\eta_{33}\rho_{23}\sigma_3^3-\eta_{33}\rho_{23}\sigma_3^3-\delta_3\eta_{13}\sigma_3^2-\delta_1\eta_{33}\sigma_3^2-\delta_1\rho_{23}\sigma_3^2+\delta_3\rho_{23}\sigma_3^2-\delta_3\eta_{13}\rho_{23}\sigma_3^2-\delta_1\eta_{33}\rho_{23}\sigma_3^2-\eta_{13}\sigma_2\sigma_3^2-\eta_{13}\eta_{33}\sigma_2\sigma_3^2+\eta_{33}\sigma_2\sigma_3^2+\eta_{13}\xi_{31}(\delta_1-\eta_{13}\sigma_3)\sigma_3^2+\eta_{13}\xi_{31}\rho_{23}(\delta_1-\eta_{13}\sigma_3)\sigma_3^2+\xi_{31}\rho_{23}(\eta_{13}\sigma_3-\delta_1)\sigma_3^2+\delta_1\delta_3\sigma_3+\delta_1\delta_3\rho_{23}\sigma_3+\delta_1\sigma_2\sigma_3-\delta_3\sigma_2\sigma_3+\delta_3\eta_{13}\sigma_2\sigma_3+\delta_1\eta_{33}\sigma_2\sigma_3-\delta_1\xi_{31}(\delta_1-\eta_{13}\sigma_3)\sigma_3-\delta_1\xi_{31}\rho_{23}(\delta_1-\eta_{13}\sigma_3)\sigma_3+\xi_{31}\sigma_2(\delta_1-\eta_{13}\sigma_3)\sigma_3+\eta_{13}\xi_{31}\sigma_2(\eta_{13}\sigma_3-\delta_1)\sigma_3-\delta_1\delta_3\sigma_2+\delta_1\xi_{31}\sigma_2(\delta_1-\eta_{13}\sigma_3)+(\delta_1-\delta_3-\sigma_2-\eta_{13}\sigma_3+\eta_{33}\sigma_3+\rho_{23}\sigma_3+\sigma_3+\xi_{31}(\delta_1-\eta_{13}\sigma_3))(\xi_{31}\delta_1^2+(-\delta_3-(\xi_{31}+1)\sigma_2+(\eta_{33}+\rho_{23}+\xi_{31}(-2\eta_{13}+\rho_{23}+1)+1)\sigma_3)\delta_1+\delta_3(\sigma_2+(\eta_{13}-\rho_{23}-1)\sigma_3)+\sigma_3((-\eta_{33}+\eta_{13}(\xi_{31}+1)-1)\sigma_2+(\xi_{31}\eta_{13}^2-(\eta_{33}+(\xi_{31}+1)(\rho_{23}+1))\eta_{13}+\rho_{23}+\eta_{33}(\rho_{23}+1))\sigma_3)))(\xi_{31}((\rho_{23}+1)\sigma_3-\sigma_2)\delta_1^2+(\delta_3(\sigma_2-(\rho_{23}+1)\sigma_3)+\sigma_3(((\xi_{31}+1)\rho_{23}+\eta_{33}(\rho_{23}+1)-2\eta_{13}\xi_{31}(\rho_{23}+1))\sigma_3-(\eta_{33}+(1-2\eta_{13})\xi_{31}+1)\sigma_2))\delta_1+\sigma_3(\delta_3((\eta_{13}(\rho_{23}+1)-\rho_{23})\sigma_3-(\eta_{13}-1)\sigma_2)+\sigma_3(\xi_{31}((\rho_{23}+1)\sigma_3-\sigma_2)\eta_{13}^2+((\eta_{33}+\xi_{31}+1)\sigma_2-((\xi_{31}+1)\rho_{23}+\eta_{33}(\rho_{23}+1))\sigma_3)\eta_{13}+\eta_{33}(\rho_{23}\sigma_3-\sigma_2)))))$

Table 3.5: The table shows the values of Δ_i for equilibrium points $\hat{O}_3, \hat{O}_1$ on 4-dimensional subspaces $x_1 = y_1 = 0$ and $x_1 = y_2 = 0$ respectively.

S	Δ_i	Values of Δ_i for the equilibrium points $\bar{O}_2$ and $\bar{O}_3$
$\bar{O}_2$	Δ_1	$\xi_{12}\,(\delta_2 - \eta_{21}\sigma_1) - \delta_1 + \delta_2 + \eta_{11}\sigma_1 - \eta_{21}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 - \sigma_3$
	Δ_2	$\delta_2^2(\sigma_1(\eta_{11}(2\xi_{12}+1) - (\xi_{12}+1)(\xi_{12}(3\eta_{21} - \rho_{31} - 1) - \rho_{31} - 1)) - (\xi_{12}+1)^2\sigma_3) + \delta_2(\sigma_1^2(2\eta_{11}((\xi_{12}+1)(\rho_{31}+1) - \eta_{21}(2\xi_{12}+1)) + (\xi_{12}+1)(-2\eta_{21}(\xi_{12}+1)(\rho_{31}+1) + 3\eta_{21}^2\xi_{12} + (\rho_{31}+1)^2) + \eta_{11}^2) - 2(\xi_{12}+1)\sigma_3\sigma_1(-\eta_{21}(\xi_{12}+1) + \eta_{11} + \rho_{31} + 1) + (\xi_{12}+1)\sigma_3^2) - \delta_1(\delta_2 + \sigma_1(-\eta_{21} + \rho_{31} + 1) - \sigma_3)(\delta_2(2\xi_{12}+1) + \sigma_1(-\eta_{21}(2\xi_{12}+1) + 2\eta_{11} + \rho_{31} + 1) - \sigma_3) + \delta_1^2(\delta_2 + \sigma_1(-\eta_{21} + \rho_{31} + 1) - \sigma_3) + \delta_2^3\xi_{12}(\xi_{12}+1) + \sigma_1(\eta_{21}^2(\xi_{12}+1)^2\sigma_1((\rho_{31}+1)\sigma_1 - \sigma_3) - \eta_{21}(\xi_{12}+1)(\sigma_3 - (\rho_{31}+1)\sigma_1)^2 + \eta_{11}(\sigma_1(\eta_{21} - \rho_{31} - 1) + \sigma_3)(\sigma_1(\eta_{21}(2\xi_{12}+1) - \rho_{31} - 1) + \sigma_3) + \eta_{21}^3(-\xi_{12})(\xi_{12}+1)\sigma_1^2 + \eta_{11}^2\sigma_1(\sigma_1(-\eta_{21} + \rho_{31} + 1) - \sigma_3) + \rho_{31}^2\sigma_1^2 + \rho_{31}\sigma_1^2 - 2\rho_{31}\sigma_1\sigma_3 + \sigma_3^2 - \sigma_1\sigma_3)$
	Δ_3	$\sigma_1(\delta_2 - \eta_{21}\sigma_1)(\delta_1 - \delta_2\xi_{12} + (\eta_{21}\xi_{12} - \eta_{11})\sigma_1)(\rho_{31}\sigma_1 - \sigma_3)(\delta_1 - \delta_2(\xi_{12}+1) - \eta_{11}\sigma_1 + \eta_{21}\sigma_1 + \eta_{21}\xi_{12}\sigma_1 - \rho_{31}\sigma_1 - \sigma_1 + \sigma_3)^2 + (\xi_{12}((\rho_{31}+1)\sigma_1 - \sigma_3)\delta_2^2 + \sigma_1((\xi_{12}+1)(\rho_{31}\sigma_1 - \sigma_3) + \eta_{11}((\rho_{31}+1)\sigma_1 - \sigma_3) - 2\eta_{21}\xi_{12}((\rho_{31}+1)\sigma_1 - \sigma_3))\delta_2 + \sigma_1^2(\eta_{21}(\eta_{21}\xi_{12}((\rho_{31}+1)\sigma_1 - \sigma_3) - (\xi_{12}+1)(\rho_{31}\sigma_1 - \sigma_3)) - \eta_{11}(-\rho_{31}\sigma_1 + \eta_{21}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3)) + \delta_1(\sigma_1(-\rho_{31}\sigma_1 + \eta_{21}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3) - \delta_2((\rho_{31}+1)\sigma_1 - \sigma_3)))(\eta_{11}\eta_{21}\sigma_1^3 - \eta_{11}\rho_{31}\sigma_1^3 + \eta_{11}\eta_{21}\rho_{31}\sigma_1^3 + \eta_{21}\rho_{31}\sigma_1^3 - \delta_2\eta_{11}\sigma_1^2 - \delta_1\eta_{21}\sigma_1^2 + \delta_1\rho_{31}\sigma_1^2 - \delta_2\rho_{31}\sigma_1^2 - \delta_2\eta_{11}\rho_{31}\sigma_1^2 - \delta_1\eta_{21}\rho_{31}\sigma_1^2 + \eta_{21}\xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_1^2 + \eta_{21}\xi_{12}\rho_{31}(\delta_2 - \eta_{21}\sigma_1)\sigma_1^2 + \xi_{12}\rho_{31}(\eta_{21}\sigma_1 - \delta_2)\sigma_1^2 + \eta_{11}\sigma_3\sigma_1^2 - \eta_{11}\eta_{21}\sigma_3\sigma_1^2 - \eta_{21}\sigma_3\sigma_1^2 + \delta_1\delta_2\sigma_1 + \delta_1\delta_2\rho_{31}\sigma_1 - \delta_2\xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_1 - \delta_2\xi_{12}\rho_{31}(\delta_2 - \eta_{21}\sigma_1)\sigma_1 - \delta_1\sigma_3\sigma_1 + \delta_2\sigma_3\sigma_1 + \delta_2\eta_{11}\sigma_3\sigma_1 + \delta_1\eta_{21}\sigma_3\sigma_1 + \xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_3\sigma_1 + \eta_{21}\xi_{12}(\eta_{21}\sigma_1 - \delta_2)\sigma_3\sigma_1 - \delta_1\delta_2\sigma_3 + \delta_2\xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_3 + (-\delta_1 + \delta_2 + \eta_{11}\sigma_1 - \eta_{21}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 + \xi_{12}(\delta_2 - \eta_{21}\sigma_1) - \sigma_3)(\xi_{12}\delta_2^2 + ((\eta_{11} + \rho_{31} + \xi_{12}(-2\eta_{21} + \rho_{31} + 1) + 1)\sigma_1 - (\xi_{12}+1)\sigma_3)\delta_2 - \delta_1(\delta_2 + (-\eta_{21} + \rho_{31} + 1)\sigma_1 - \sigma_3) - \sigma_1(-\xi_{12}\sigma_1\eta_{21}^2 + (\xi_{12}+1)((\rho_{31}+1)\sigma_1 - \sigma_3)\eta_{21} - \rho_{31}\sigma_1 + \sigma_3 + \eta_{11}((\eta_{21} - \rho_{31} - 1)\sigma_1 + \sigma_3))))$
	Δ_4	$\sigma_1(\delta_2 - \eta_{21}\sigma_1)(-\delta_1 + \delta_2\xi_{12} + (\eta_{11} - \eta_{21}\xi_{12})\sigma_1)(\rho_{31}\sigma_1 - \sigma_3)(\sigma_1(\delta_2 - \eta_{21}\sigma_1)(\delta_1 - \delta_2\xi_{12} + (\eta_{21}\xi_{12} - \eta_{11})\sigma_1)(\rho_{31}\sigma_1 - \sigma_3)(\delta_1 - \delta_2(\xi_{12}+1) - \eta_{11}\sigma_1 + \eta_{21}\sigma_1 + \eta_{21}\xi_{12}\sigma_1 - \rho_{31}\sigma_1 - \sigma_1 + \sigma_3)^2 + (\xi_{12}((\rho_{31}+1)\sigma_1 - \sigma_3)\delta_2^2 + \sigma_1((\xi_{12}+1)(\rho_{31}\sigma_1 - \sigma_3) + \eta_{11}((\rho_{31}+1)\sigma_1 - \sigma_3) - 2\eta_{21}\xi_{12}((\rho_{31}+1)\sigma_1 - \sigma_3))\delta_2 + \sigma_1^2(\eta_{21}(\eta_{21}\xi_{12}((\rho_{31}+1)\sigma_1 - \sigma_3) - (\xi_{12}+1)(\rho_{31}\sigma_1 - \sigma_3)) - \eta_{11}(-\rho_{31}\sigma_1 + \eta_{21}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3)) + \delta_1(\sigma_1(-\rho_{31}\sigma_1 + \eta_{21}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3) - \delta_2((\rho_{31}+1)\sigma_1 - \sigma_3)))(\eta_{11}\eta_{21}\sigma_1^3 - \eta_{11}\rho_{31}\sigma_1^3 + \eta_{11}\eta_{21}\rho_{31}\sigma_1^3 + \eta_{21}\rho_{31}\sigma_1^3 - \delta_2\eta_{11}\sigma_1^2 - \delta_1\eta_{21}\sigma_1^2 + \delta_1\rho_{31}\sigma_1^2 - \delta_2\rho_{31}\sigma_1^2 - \delta_2\eta_{11}\rho_{31}\sigma_1^2 - \delta_1\eta_{21}\rho_{31}\sigma_1^2 + \eta_{21}\xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_1^2 + \eta_{21}\xi_{12}\rho_{31}(\delta_2 - \eta_{21}\sigma_1)\sigma_1^2 + \xi_{12}\rho_{31}(\eta_{21}\sigma_1 - \delta_2)\sigma_1^2 + \eta_{11}\sigma_3\sigma_1^2 - \eta_{11}\eta_{21}\sigma_3\sigma_1^2 - \eta_{21}\sigma_3\sigma_1^2 + \delta_1\delta_2\sigma_1 + \delta_1\delta_2\rho_{31}\sigma_1 - \delta_2\xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_1 - \delta_2\xi_{12}\rho_{31}(\delta_2 - \eta_{21}\sigma_1)\sigma_1 - \delta_1\sigma_3\sigma_1 + \delta_2\sigma_3\sigma_1 + \delta_2\eta_{11}\sigma_3\sigma_1 + \delta_1\eta_{21}\sigma_3\sigma_1 + \xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_3\sigma_1 + \eta_{21}\xi_{12}(\eta_{21}\sigma_1 - \delta_2)\sigma_3\sigma_1 - \delta_1\delta_2\sigma_3 + \delta_2\xi_{12}(\delta_2 - \eta_{21}\sigma_1)\sigma_3 + (-\delta_1 + \delta_2 + \eta_{11}\sigma_1 - \eta_{21}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 + \xi_{12}(\delta_2 - \eta_{21}\sigma_1) - \sigma_3)(\xi_{12}\delta_2^2 + ((\eta_{11} + \rho_{31} + \xi_{12}(-2\eta_{21} + \rho_{31} + 1) + 1)\sigma_1 - (\xi_{12}+1)\sigma_3)\delta_2 - \delta_1(\delta_2 + (-\eta_{21} + \rho_{31} + 1)\sigma_1 - \sigma_3) - \sigma_1(-\xi_{12}\sigma_1\eta_{21}^2 + (\xi_{12}+1)((\rho_{31}+1)\sigma_1 - \sigma_3)\eta_{21} - \rho_{31}\sigma_1 + \sigma_3 + \eta_{11}((\eta_{21} - \rho_{31} - 1)\sigma_1 + \sigma_3)))))$
$\bar{O}_3$	Δ_1	$\xi_{23}\,(\delta_3 - \eta_{31}\sigma_1) - \delta_2 + \delta_3 + \eta_{21}\sigma_1 - \eta_{31}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 - \sigma_3$
	Δ_2	$\delta_3^2(\sigma_1(\eta_{21}(2\xi_{23}+1) - (\xi_{23}+1)(\xi_{23}(3\eta_{31} - \rho_{31} - 1) - \rho_{31} - 1)) - (\xi_{23}+1)^2\sigma_3) + \delta_3(\sigma_1^2(2\eta_{21}((\xi_{23}+1)(\rho_{31}+1) - \eta_{31}(2\xi_{23}+1)) + (\xi_{23}+1)(-2\eta_{31}(\xi_{23}+1)(\rho_{31}+1) + 3\eta_{31}^2\xi_{23} + (\rho_{31}+1)^2) + \eta_{21}^2) - 2(\xi_{23}+1)\sigma_3\sigma_1(-\eta_{31}(\xi_{23}+1) + \eta_{21} + \rho_{31} + 1) + (\xi_{23}+1)\sigma_3^2) - \delta_2(\delta_3 + \sigma_1(-\eta_{31} + \rho_{31} + 1) - \sigma_3)(\delta_3(2\xi_{23}+1) + \sigma_1(-\eta_{31}(2\xi_{23}+1) + 2\eta_{21} + \rho_{31} + 1) - \sigma_3) + \delta_2^2(\delta_3 + \sigma_1(-\eta_{31} + \rho_{31} + 1) - \sigma_3) + \delta_3^3\xi_{23}(\xi_{23}+1) + \sigma_1(\eta_{31}^2(\xi_{23}+1)^2\sigma_1((\rho_{31}+1)\sigma_1 - \sigma_3) - \eta_{31}(\xi_{23}+1)(\sigma_3 - (\rho_{31}+1)\sigma_1)^2 + \eta_{21}(\sigma_1(\eta_{31} - \rho_{31} - 1) + \sigma_3)(\sigma_1(\eta_{31}(2\xi_{23}+1) - \rho_{31} - 1) + \sigma_3) + \eta_{31}^3(-\xi_{23})(\xi_{23}+1)\sigma_1^2 + \eta_{21}^2\sigma_1(\sigma_1(-\eta_{31} + \rho_{31} + 1) - \sigma_3) + \rho_{31}^2\sigma_1^2 + \rho_{31}\sigma_1^2 - 2\rho_{31}\sigma_1\sigma_3 + \sigma_3^2 - \sigma_1\sigma_3)$
	Δ_3	$\sigma_1(\delta_3 - \eta_{31}\sigma_1)(\delta_2 - \delta_3\xi_{23} + (\eta_{31}\xi_{23} - \eta_{21})\sigma_1)(\rho_{31}\sigma_1 - \sigma_3)(\delta_2 - \delta_3(\xi_{23}+1) - \eta_{21}\sigma_1 + \eta_{31}\sigma_1 + \eta_{31}\xi_{23}\sigma_1 - \rho_{31}\sigma_1 - \sigma_1 + \sigma_3)^2 + (\xi_{23}((\rho_{31}+1)\sigma_1 - \sigma_3)\delta_3^2 + \sigma_1((\xi_{23}+1)(\rho_{31}\sigma_1 - \sigma_3) + \eta_{21}((\rho_{31}+1)\sigma_1 - \sigma_3) - 2\eta_{31}\xi_{23}((\rho_{31}+1)\sigma_1 - \sigma_3))\delta_3 + \sigma_1^2(\eta_{31}(\eta_{31}\xi_{23}((\rho_{31}+1)\sigma_1 - \sigma_3) - (\xi_{23}+1)(\rho_{31}\sigma_1 - \sigma_3)) - \eta_{21}(-\rho_{31}\sigma_1 + \eta_{31}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3)) + \delta_2(\sigma_1(-\rho_{31}\sigma_1 + \eta_{31}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3) - \delta_3((\rho_{31}+1)\sigma_1 - \sigma_3)))(\eta_{21}\eta_{31}\sigma_1^3 - \eta_{21}\rho_{31}\sigma_1^3 + \eta_{21}\eta_{31}\rho_{31}\sigma_1^3 + \eta_{31}\rho_{31}\sigma_1^3 - \delta_3\eta_{21}\sigma_1^2 - \delta_2\eta_{31}\sigma_1^2 + \delta_2\rho_{31}\sigma_1^2 - \delta_3\rho_{31}\sigma_1^2 - \delta_3\eta_{21}\rho_{31}\sigma_1^2 - \delta_2\eta_{31}\rho_{31}\sigma_1^2 + \eta_{31}\xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_1^2 + \eta_{31}\xi_{23}\rho_{31}(\delta_3 - \eta_{31}\sigma_1)\sigma_1^2 + \xi_{23}\rho_{31}(\eta_{31}\sigma_1 - \delta_3)\sigma_1^2 + \eta_{21}\sigma_3\sigma_1^2 - \eta_{21}\eta_{31}\sigma_3\sigma_1^2 - \eta_{31}\sigma_3\sigma_1^2 + \delta_2\delta_3\sigma_1 + \delta_2\delta_3\rho_{31}\sigma_1 - \delta_3\xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_1 - \delta_3\xi_{23}\rho_{31}(\delta_3 - \eta_{31}\sigma_1)\sigma_1 - \delta_2\sigma_3\sigma_1 + \delta_3\sigma_3\sigma_1 + \delta_3\eta_{21}\sigma_3\sigma_1 + \delta_2\eta_{31}\sigma_3\sigma_1 + \xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_3\sigma_1 + \eta_{31}\xi_{23}(\eta_{31}\sigma_1 - \delta_3)\sigma_3\sigma_1 - \delta_2\delta_3\sigma_3 + \delta_3\xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_3 + (-\delta_2 + \delta_3 + \eta_{21}\sigma_1 - \eta_{31}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 + \xi_{23}(\delta_3 - \eta_{31}\sigma_1) - \sigma_3)(\xi_{23}\delta_3^2 + ((\eta_{21} + \rho_{31} + \xi_{23}(-2\eta_{31} + \rho_{31} + 1) + 1)\sigma_1 - (\xi_{23}+1)\sigma_3)\delta_3 - \delta_2(\delta_3 + (-\eta_{31} + \rho_{31} + 1)\sigma_1 - \sigma_3) - \sigma_1(-\xi_{23}\sigma_1\eta_{31}^2 + (\xi_{23}+1)((\rho_{31}+1)\sigma_1 - \sigma_3)\eta_{31} - \rho_{31}\sigma_1 + \sigma_3 + \eta_{21}((\eta_{31} - \rho_{31} - 1)\sigma_1 + \sigma_3))))$
	Δ_4	$\sigma_1(\delta_3 - \eta_{31}\sigma_1)(-\delta_2 + \delta_3\xi_{23} + (\eta_{21} - \eta_{31}\xi_{23})\sigma_1)(\rho_{31}\sigma_1 - \sigma_3)(\sigma_1(\delta_3 - \eta_{31}\sigma_1)(\delta_2 - \delta_3\xi_{23} + (\eta_{31}\xi_{23} - \eta_{21})\sigma_1)(\rho_{31}\sigma_1 - \sigma_3)(\delta_2 - \delta_3(\xi_{23}+1) - \eta_{21}\sigma_1 + \eta_{31}\sigma_1 + \eta_{31}\xi_{23}\sigma_1 - \rho_{31}\sigma_1 - \sigma_1 + \sigma_3)^2 + (\xi_{23}((\rho_{31}+1)\sigma_1 - \sigma_3)\delta_3^2 + \sigma_1((\xi_{23}+1)(\rho_{31}\sigma_1 - \sigma_3) + \eta_{21}((\rho_{31}+1)\sigma_1 - \sigma_3) - 2\eta_{31}\xi_{23}((\rho_{31}+1)\sigma_1 - \sigma_3))\delta_3 + \sigma_1^2(\eta_{31}(\eta_{31}\xi_{23}((\rho_{31}+1)\sigma_1 - \sigma_3) - (\xi_{23}+1)(\rho_{31}\sigma_1 - \sigma_3)) - \eta_{21}(-\rho_{31}\sigma_1 + \eta_{31}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3)) + \delta_2(\sigma_1(-\rho_{31}\sigma_1 + \eta_{31}((\rho_{31}+1)\sigma_1 - \sigma_3) + \sigma_3) - \delta_3((\rho_{31}+1)\sigma_1 - \sigma_3)))(\eta_{21}\eta_{31}\sigma_1^3 - \eta_{21}\rho_{31}\sigma_1^3 + \eta_{21}\eta_{31}\rho_{31}\sigma_1^3 + \eta_{31}\rho_{31}\sigma_1^3 - \delta_3\eta_{21}\sigma_1^2 - \delta_2\eta_{31}\sigma_1^2 + \delta_2\rho_{31}\sigma_1^2 - \delta_3\rho_{31}\sigma_1^2 - \delta_3\eta_{21}\rho_{31}\sigma_1^2 - \delta_2\eta_{31}\rho_{31}\sigma_1^2 + \eta_{31}\xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_1^2 + \eta_{31}\xi_{23}\rho_{31}(\delta_3 - \eta_{31}\sigma_1)\sigma_1^2 + \xi_{23}\rho_{31}(\eta_{31}\sigma_1 - \delta_3)\sigma_1^2 + \eta_{21}\sigma_3\sigma_1^2 - \eta_{21}\eta_{31}\sigma_3\sigma_1^2 - \eta_{31}\sigma_3\sigma_1^2 + \delta_2\delta_3\sigma_1 + \delta_2\delta_3\rho_{31}\sigma_1 - \delta_3\xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_1 - \delta_3\xi_{23}\rho_{31}(\delta_3 - \eta_{31}\sigma_1)\sigma_1 - \delta_2\sigma_3\sigma_1 + \delta_3\sigma_3\sigma_1 + \delta_3\eta_{21}\sigma_3\sigma_1 + \delta_2\eta_{31}\sigma_3\sigma_1 + \xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_3\sigma_1 + \eta_{31}\xi_{23}(\eta_{31}\sigma_1 - \delta_3)\sigma_3\sigma_1 - \delta_2\delta_3\sigma_3 + \delta_3\xi_{23}(\delta_3 - \eta_{31}\sigma_1)\sigma_3 + (-\delta_2 + \delta_3 + \eta_{21}\sigma_1 - \eta_{31}\sigma_1 + \rho_{31}\sigma_1 + \sigma_1 + \xi_{23}(\delta_3 - \eta_{31}\sigma_1) - \sigma_3)(\xi_{23}\delta_3^2 + ((\eta_{21} + \rho_{31} + \xi_{23}(-2\eta_{31} + \rho_{31} + 1) + 1)\sigma_1 - (\xi_{23}+1)\sigma_3)\delta_3 - \delta_2(\delta_3 + (-\eta_{31} + \rho_{31} + 1)\sigma_1 - \sigma_3) - \sigma_1(-\xi_{23}\sigma_1\eta_{31}^2 + (\xi_{23}+1)((\rho_{31}+1)\sigma_1 - \sigma_3)\eta_{31} - \rho_{31}\sigma_1 + \sigma_3 + \eta_{21}((\eta_{31} - \rho_{31} - 1)\sigma_1 + \sigma_3)))))$

Table 3.6: The table shows the values of Δ_i for equilibrium points $\bar{O}_2, \bar{O}_3$ on 4-dimensional subspaces $x_2 = y_3 = 0$ and $x_2 = y_2 = 0$ respectively.

Chapter 4

Numerical results

The goal of this chapter is to confirm theorical results by numerical experiments. First, we show how heteroclinic trajectories belonging to the heteroclinic network behave. Secondly, we present a picture of the trajectories on the heteroclinic attractor.

4.1 Heteroclnic Network

Assume that $\sigma_1 = 6, \sigma_2 = 7, \sigma_3 = 8, \delta_1 = 10, \delta_2 = 15, \delta_3 = 20, \eta_{11} = 0.1, \eta_{12} = 0.2, \eta_{13} = 0.3, \eta_{21} = 0.1, \eta_{22} = 0.2, \eta_{23} = 0.3, \eta_{31} = 0.1, \eta_{32} = 0.2, \eta_{33} = 0.3$. Then parameters in the table 4.1 satisfy the conditions of Therem 3.4.1.

Parameters	**Values of parameters**				
ρ_{12}	3.64512	3.39916	3.79064	1.72127	4.52017
ρ_{13}	0.136261	0.134727	0.269263	0.373726	0.265443
ρ_{21}	0.139889	0.0468083	0.0174621	0.192648	0.0348452
ρ_{23}	2.39146	3.09165	1.25919	1.44744	1.51453
ρ_{31}	2.1775	5.73921	2.29126	2.29559	2.79281
ρ_{32}	0.225068	0.0577013	0.19216	0.338097	0.136058
ξ_{12}	2.75417	1.56495	3.18986	2.37243	4.17485
ξ_{13}	0.0369821	0.251189	0.138018	0.268559	0.143376
ξ_{21}	0.027865	0.19193	0.151999	0.199366	0.0671996
ξ_{23}	3.35939	2.44048	2.69774	3.55245	4.93359
ξ_{31}	6.94691	2.71107	5.53754	3.12758	5.65943
ξ_{32}	0.171463	0.118564	0.311079	0.0130082	0.0984532

Table 4.1: ρ_{ij} and ξ_{ij} values.

Let us choose for example values of parameters which are in the second column in the table 4.1. Therefore for these parameters, the master-slave is the following

$$\begin{cases} \dot{x_1} = x_1(6 - x_1 - 3.64512x_2 - 0.136261x_3) \\ \dot{x_2} = x_2(7 - x_2 - 0.139889x_1 - 2.39146x_3) \\ \dot{x_3} = x_3(8 - x_3 - 2.1775x_1 - \ 0.225068x_2) \\ \dot{y_1} = y_1(10 - y_1 - 2.75417y_2 - 0.0369821y_3 - 0.1x_1 - 0.2x_2 - 0.3x_3) \\ \dot{y_2} = y_2(15 - \ \ y_2 \ \ 0.027865y_1 - \ \ 3.35939y_3 - 0.1x_1 - 0.2x_2 - 0.3x_3) \\ \dot{y_3} = y_3(20 - \ \ y_3 \ \ 6.94691y_1 - \ \ 0.171463y_2 - 0.1x_1 - 0.2x_2 - 0.3x_3) \end{cases} \tag{4.1}$$

Now computing the system (4.1) for initial point $p_0 = (4, 4, 4, 6, 7, 5)$ we have the following portraits in the phase space. First, let us show the

4.1.1 Trajectories of heteroclinic cycles

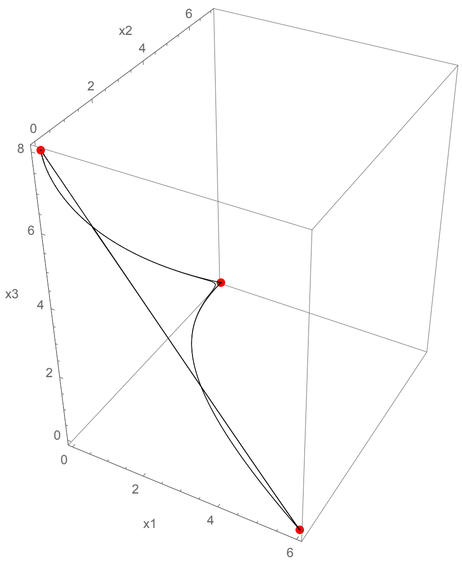

Figure 4.1: Heteroclinic cycle of master system.

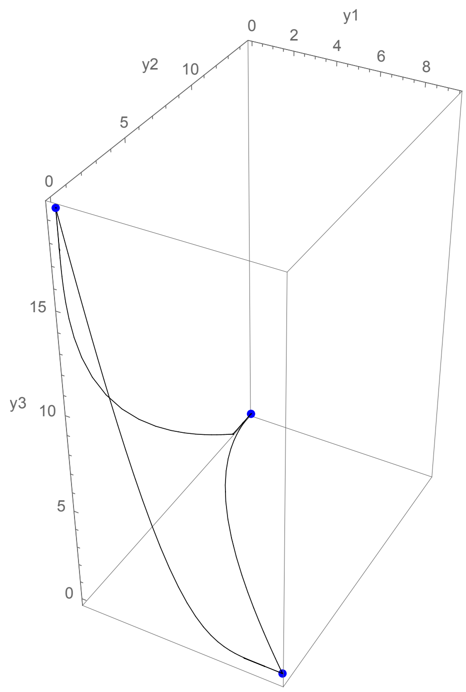

Figure 4.2: Heteroclinic cycle $\bar{\Gamma}$ of master-slave system.

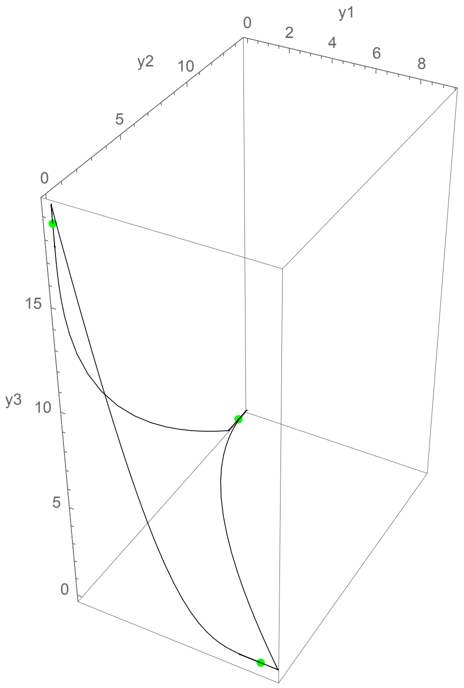

Figure 4.3: Heteroclinic cycle $\dot{\Gamma}$ of master-slave system.

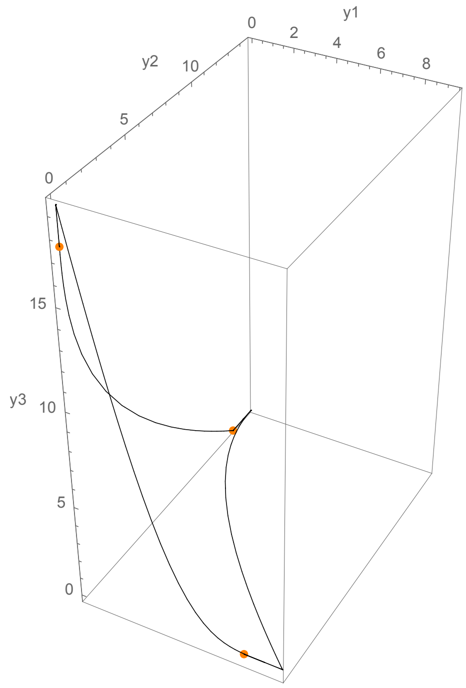

Figure 4.4: Heteroclinic cycle $\hat{\Gamma}$ of master-slave system.

Now let us show the

4.1.2 Trajectories joining different cycles

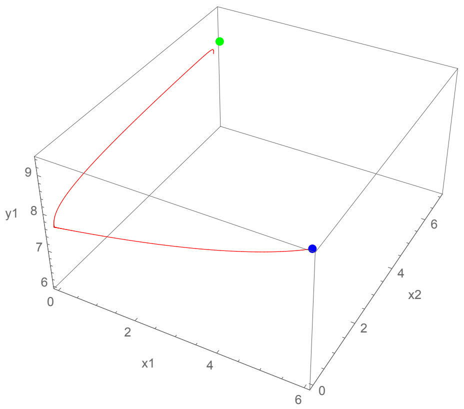

Figure 4.5: Heteroclinic connection between $\bar{O}_1$ and $\dot{O}_1$.

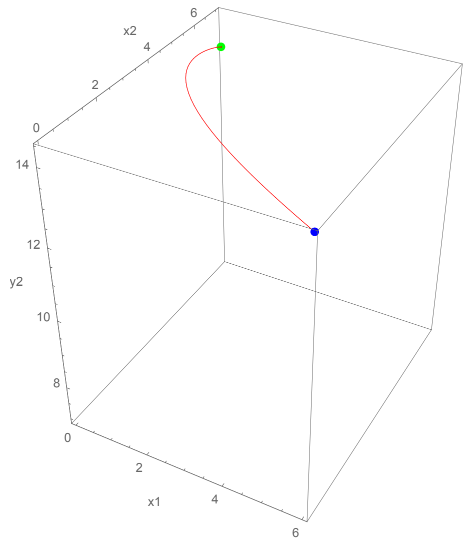

Figure 4.6: Heteroclinic connection between $\bar{O}_2$ and $\dot{O}_2$.

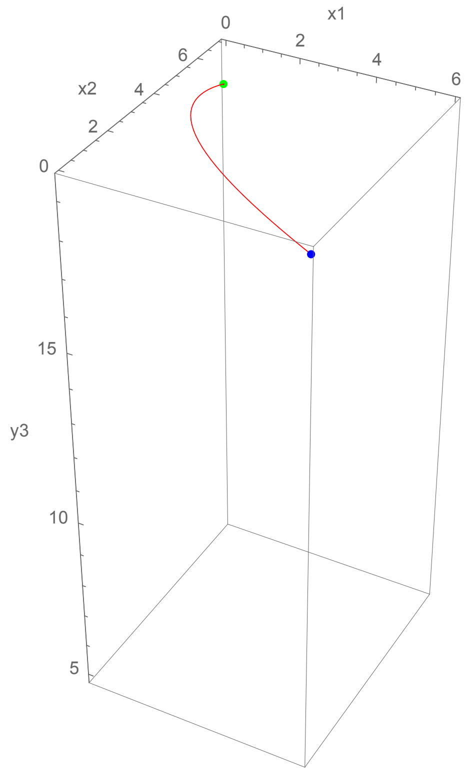

Figure 4.7: Heteroclinic connection between $\bar{O}_3$ and $\dot{O}_3$.

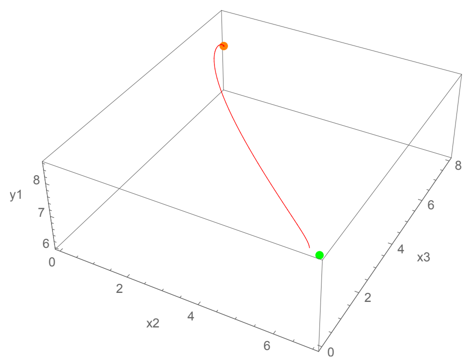

Figure 4.8: Heteroclinic connection between $\dot{O}_1$ and $\hat{O}_1$.

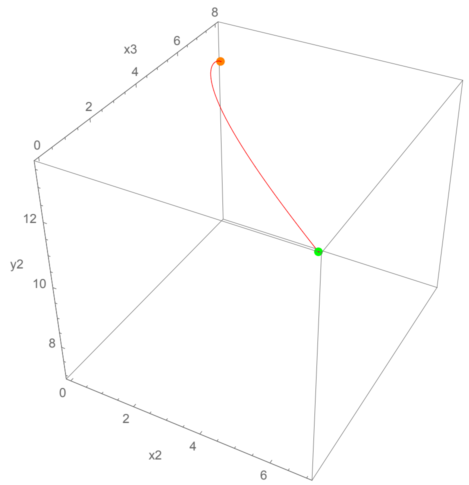

Figure 4.9: Heteroclinic connection between $\dot{O}_2$ and $\hat{O}_2$.

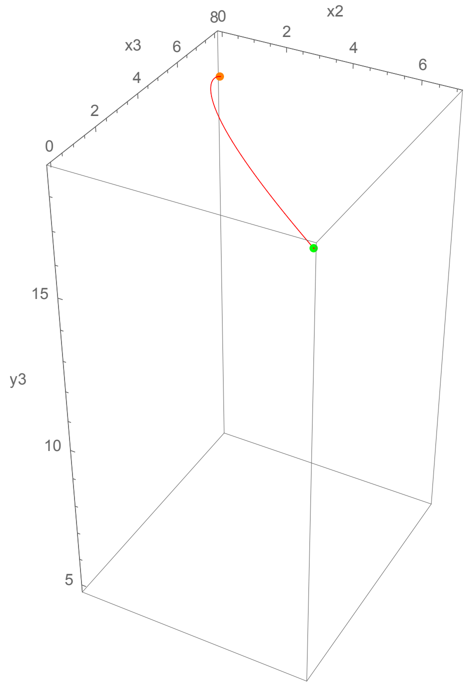

Figure 4.10: Heteroclinic connection between $\dot{O}_3$ and $\hat{O}_3$.

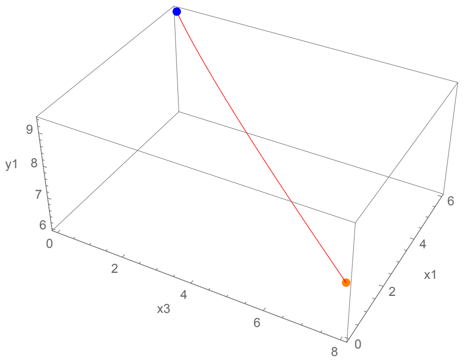

Figure 4.11: Heteroclinic connection between $\hat{O}_1$ and $\bar{O}_1$.

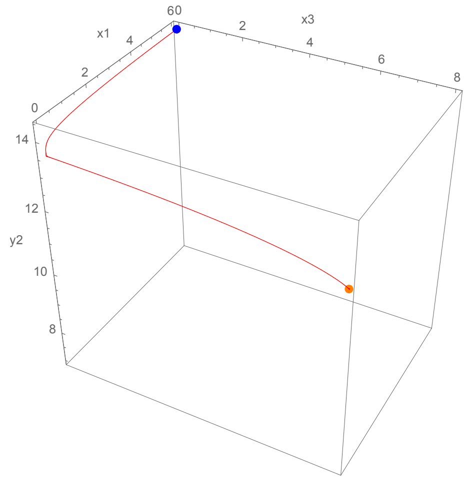

Figure 4.12: Heteroclinic connection between $\hat{O}_2$ and $\bar{O}_2$.

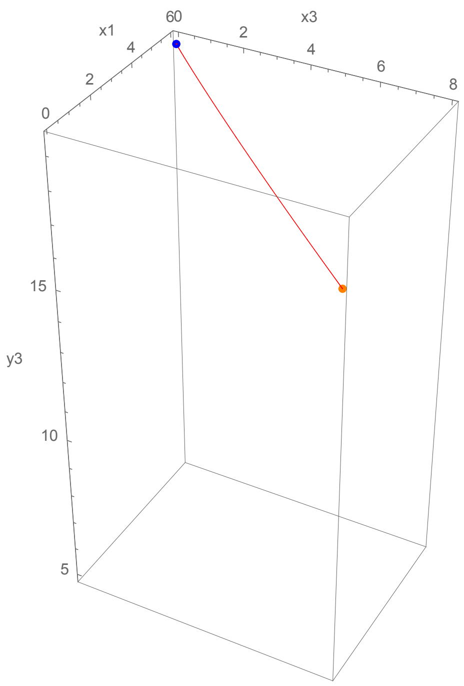

Figure 4.13: Heteroclinic connection between $\hat{O}_3$ and $\bar{O}_3$.

One can see that when we compute the system (3.3)-(3.4) for parameters which satisfy inequalites of Teorem 3.4.1, we obtain every heteroclinic connection of the heteroclinic network Γ.

4.2 Heteroclinic attractor

Here we check numerically that pieces 2-dimensional unstable manifolds of saddle equilibrium points form a two-dimensional surface consisted of topological rectangules. Let us consider, first the point $\bar{O}_1$. Its unstable manifold in linear approximation has the form of the plane

$$(x,y) = (x_0,y_0) + \alpha e_1 + \beta e_2 \tag{4.2}$$

where (x_0,y_0) are the coordinates of $\bar{O}_1$ and e_1, e_2 are the eigenvectors corresponding to positive eigenvalues.

We choose initial points belonging to plane (4.2) for α and β small and show that trajectories going through all of them tend to $\dot{O}_2$.

Let us consider $\sigma_1 = 6, \sigma_2 = 7, \sigma_3 = 8, \delta_1 = 10, \delta_2 = 15, \delta_3 = 20, \rho_{12} = 1.72127, \rho_{13} = 0.373726, \rho_{21} = 0.192648, \rho_{23} = 1.44744, \rho_{31} = 2.29559, \rho_{32} = 0.338097, \xi_{12} = 2.37243$ $, \xi_{13} = 0.268559, \xi_{21} = 0.199366, \xi_{23} = 3.55245, \xi_{31} = 3.12758, \xi_{32} = 0.0130082, \eta_{11} = \eta_{12} = \eta_{13} = \eta_{21} = \eta_{22} = \eta_{23} = \eta_{31} = \eta_{32} = \eta_{33} = 0.001$. One can check that these parameters satisfy conditions of Theorem 3.4.1. It system is the following

$$\begin{cases} \dot{x_1} = x_1(6 - x_1 - 1.72127x_2 - 0.373726x_3) \\ \dot{x_2} = x_2(7 - x_2 - 0.192648x_1 - 1.44744x_3) \\ \dot{x_3} = x_3(8 - x_3 - 2.29559x_1 - 0.338097x_2) \\ \dot{y_1} = y_1(10 - y_1 - 2.37243y_2 - 0.268559y_3 - 0.001x_1 - 0.001x_2 - 0.001x_3) \\ \dot{y_2} = y_2(15 - y_2 - 0.199366y_1 - 3.55245y_3 - 0.001x_1 - 0.001x_2 - 0.001x_3) \\ \dot{y_3} = y_3(20 - y_3 - 3.12758y_1 - 0.0130082y_2 - 0.001x_1 - 0.001x_2 - 0.001x_3) \end{cases} \tag{4.3}$$

Every trajectory of heteroclinic network was obtained for this system as in the previous section. Consider the invariant 4-dimensional subspace $x_3 = y_3 = 0$ of system (4.3). It subspace has de form

$$\begin{cases} \dot{x_1} = x_1(6 - x_1 - 1.72127x_2) \\ \dot{x_2} = x_2(7 - x_2 - 0.192648x_1) \\ \dot{y_1} = y_1(10 - y_1 - 2.37243y_2 - 0.001x_1 - 0.001x_2) \\ \dot{y_2} = y_2(15 - y_2 - 0.199366y_1 - 0.001x_1 - 0.001x_2) \end{cases} \tag{4.4}$$

Interesting equilibrium points of system (4.4) are $\bar{O}_1 = (6,0,9.994,0), \bar{O}_2 = (6,0,0,14.994), \dot{O}_1 = (0,7,9.993,0)$ and $\dot{O}_2 = (0,7,0,14.993)$. Eigenvalues of the Jacobian matrix of system (4.4) evaluated at $\bar{O}_1$ are $\lambda_1 = 13.0015, \lambda_2 = 5.84411, \lambda_3 = -9.994, \lambda_4 = -6$ and $e_1 = (0,0,-0.717841,0.696208)$, $e_2 = (-0.657207, 0.75371, -0.0000608945, 0)$ are the corresponding eigenvectors to λ_1 and λ_2 respectively. Eigenvalues of the Jacobian matrix of system (4.4) evaluated at $\bar{O}_2$ and $\dot{O}_1$ are $b_1 = -25.5782, b_2 = 5.84411, b_3 = -14.994, b_4 = -6$ and $m_1 = 13.0007, m_2 = -7, m_3 = -9.993, m_4 = -6.04889$ respectively, therefore $dimW^u(\bar{O}_2) = dimW^u(\dot{O}_1) = 1$. Finally the eigenvalues of the Jacobian matrix of system (4.4) evaluated at $\dot{O}_2$ are $r_1 = -25.5768, r_2 = -14.993, r_3 = -7$ and $r_4 = -6.04889$, so $\dot{O}_2$ is a stable node.

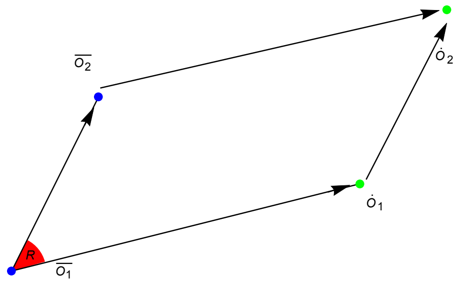

Figure 4.14: Heteroclinic connections on subspace (4.4).

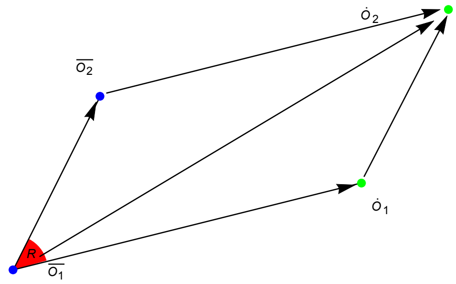

Figure 4.15: Trajectories of points inside of region R go to $\dot{O}_2$.

Let us choose $\alpha + \beta = \epsilon = 0.01$. So, we have for example the following 9 initial points.

α_i	β_i	x_{0i}	$x_{0i} = +\alpha_i e_1 + \beta_i e_2$
0.001	0.009	x_{01}	(5.99409, 0.00678339, 9.99328, 0.000696208)
0.002	0.008	x_{02}	(5.99474, 0.00602968, 9.99256, 0.00139242)
0.003	0.007	x_{03}	(5.9954, 0.00527597, 9.99185, 0.00208862)
0.004	0.006	x_{04}	(5.99606, 0.00452226, 9.99113, 0.00278483)
0.005	0.005	x_{05}	(5.99671, 0.00376855, 9.99041, 0.00348104)
0.006	0.004	x_{06}	(5.99737, 0.00301484, 9.98969, 0.00417725)
0.007	0.003	x_{07}	(5.99803, 0.00226113, 9.98897, 0.00487346)
0.008	0.002	x_{08}	(5.99869, 0.00150742, 9.98826, 0.00556966)
0.009	0.001	x_{09}	(5.99934, 0.00075371, 9.98754, 0.00626587)

Table 4.2: Initial points x_{0i} inside of R.

One can see that the points in the table 4.2 satisfy that $x_{0i} \in R, i = 1, ..., 9$. Computing the system (4.4) for each of the points x_{0i} in the table 4.2, we obtain

x_{0i}	Final point of trajectoy of x_{0i} at time $t = 100$
x_{01}	$(1.19323x10^{-21}, 7, -2.14585x10^{-27}, 14.993)$
x_{02}	$(2.83636x10^{-21}, 7, 8.06246x10^{-27}, 14.993)$
x_{03}	$(4.87317x10^{-22}, 7, -3.70539x10^{-29}, 14.993)$
x_{04}	$(3.22109x10^{-20}, 7, 5.52008x10^{-25}, 14.993)$
x_{05}	$(4.40345x10^{-21}, 7, -7.56563x10^{-27}, 14.993)$
x_{06}	$(8.68163x10^{-21}, 7, -6.01977x10^{-27}, 14.993)$
x_{07}	$(9.95544x10^{-21}, 7, -4.5837x10^{-24}, 14.993)$
x_{08}	$(4.97699x10^{-22}, 7, 5.29206x10^{-27}, 14.993)$
x_{09}	$(3.17621x10^{-21}, 7, -3.84468x10^{-26}, 14.993)$

Table 4.3: Trajectories of initial points inside of the region R are very closed to the equilibrium point $\dot{O}_2 = (0, 7, 0, 14.993)$ at time $t = 100$.

Therefore the corresponding trajectory to x_{0i} goes to $\dot{O}_2$, $i = 1, ..., 9$. Similarly we obtained for the 4-dimensional invariant subspaces $x_3 = y_1 = 0, x_3 = y_2 = 0, x_1 =$

$y_3 = 0, x_1 = y_1 = 0, x_1 = y_2 = 0, x_2 = y_3 = 0, x_2 = y_1 = 0$ and $x_2 = y_2 = 0$, the results which are shown in figures 4.16 4.17,4.18,4.19,4.20,4.21,4.22 and 4.23 respectively. These same results were obtained for a system where $\eta \equiv 0$.

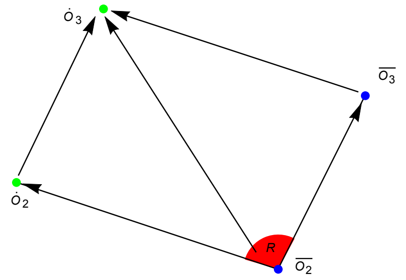

Figure 4.16: Trajectories of points inside of region R go to $\dot{O}_3$.

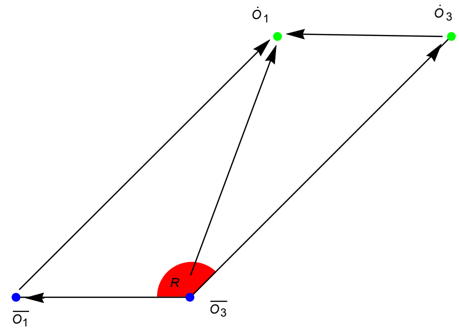

Figure 4.17: Trajectories of points inside of region R go to $\dot{O}_1$.

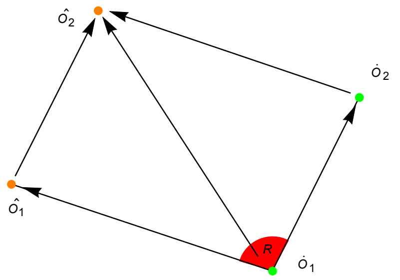

Figure 4.18: Trajectories of points inside of region R go to $\hat{O}_2$.

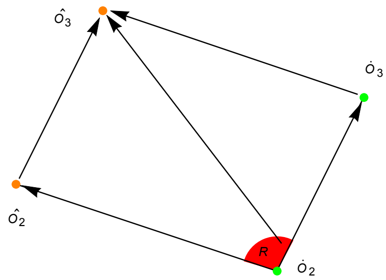

Figure 4.19: Trajectories of points inside of region R go to $\hat{O}_3$.

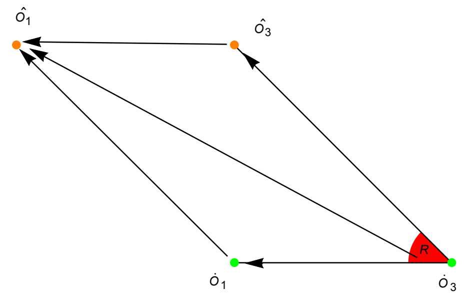

Figure 4.20: Trajectories of points inside of region R go to $\hat{O}_1$.

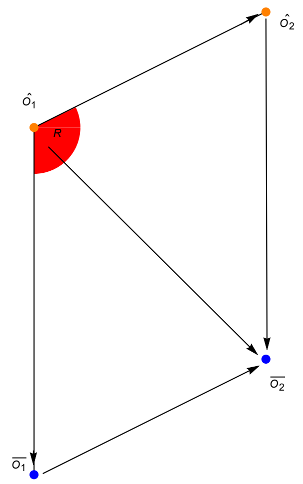

Figure 4.21: Trajectories of points inside of region R go to $\bar{O}_2$.

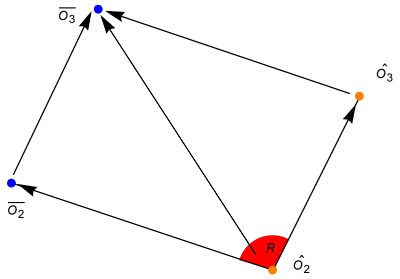

Figure 4.22: Trajectories of points inside of region R go to $\bar{O}_3$.

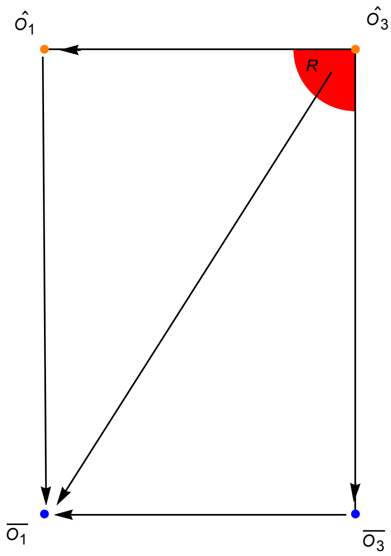

Figure 4.23: Trajectories of points inside of region R go to $\bar{O}_1$.

Thus, we confirmed the existence of topological rectangles union of which form a topological two-dimensional torus. This torus could be an attractor to show it numerically we choose initial points randomly in a neighborhood of the torus and show that corresponding trajectories go to it as $t \to +\infty$.

First let us choose randomly, for instance, the following 20 initial points which are in a neighborhood of this torus. These initial points are shown in the table 4.4.

x_{0i}	
x_{01}	$(6.41264, 9.3393, 8.90569, 11.1921, 17.4231, 22.0908)$
x_{02}	$(10.0606, 7.82248, 10.0456, 12.0855, 17.6599, 22.0487)$
x_{03}	$(10.4133, 11.4697, 8.14363, 12.4614, 15.5398, 22.3685)$
x_{04}	$(8.40094, 9.74301, 12.0417, 12.7016, 15.1959, 22.2957)$
x_{05}	$(8.72313, 11.3684, 8.6514, 11.176, 17.1261, 23.7437)$
x_{06}	$(9.35894, 8.47259, 8.06352, 14.5503, 18.2687, 24.2453)$
x_{07}	$(7.21408, 9.18421, 9.03217, 12.5283, 18.9746, 23.4664)$
x_{08}	$(9.50755, 7.4743, 10.0736, 13.8237, 17.2361, 23.3234)$
x_{09}	$(6.0644, 8.17875, 10.8694, 14.1598, 18.0462, 24.2367)$
x_{010}	$(8.13248, 9.88147, 12.3621, 11.5941, 18.0045, 22.7155)$
x_{011}	$(10.9199, 8.87088, 11.8605, 14.0527, 15.4936, 21.5978)$
x_{012}	$(7.98274, 11.6366, 8.42566, 13.6998, 19.7064, 23.7039)$
x_{013}	$(9.20681, 9.62526, 8.28053, 14.6364, 15.6793, 20.0888)$
x_{014}	$(9.36108, 7.75119, 9.77988, 12.8574, 19.5286, 24.4935)$
x_{015}	$(10.8242, 7.19994, 9.64262, 14.329, 15.0395, 21.0991)$
x_{016}	$(8.58968, 10.2095, 12.7158, 12.1666, 15.1383, 22.3791)$
x_{017}	$(9.24342, 8.96311, 8.76119, 13.2471, 17.5017, 20.4649)$
x_{018}	$(6.42592, 7.81848, 12.8981, 12.7988, 19.4375, 23.1818)$
x_{019}	$(7.90655, 9.71962, 8.05679, 13.4287, 17.5346, 22.8958)$
x_{020}	$(7.09771, 7.98509, 8.63374, 11.2042, 19.0657, 22.5987)$

Table 4.4: Initial points in a neighborhood of torus.

Now computing the master-slave system (4.3) for every initial points of the table 4.4, we obtain the results which are shown in the table 4.5.

x_{0i}	Final point of trajectoy of x_{0i} at time $t = 100$
x_{01}	$(4.66109x10^{-13}, 9.8009x10^{-14}, 8, 0.0000648813, 0, 19.9918)$
x_{02}	$(5.70365x10^{-24}, 0.40845, 7.63524, 9.99196, 0, 1.84443x10^{-71})$
x_{03}	$(1.78512x10^{-11}, 1.5832x10^{-17}, 8, 9.992, 0, 2.84259x10^{-202})$
x_{04}	$(0.0000613748, 6.99992, 2.95197x10^{-19}, 9.993, 0, 8.59602x10^{-236})$
x_{05}	$(1.78737x10^{-13}, 7, 7.44608x10^{-12}, 8.27526x10^{-22}, 0, 19.993)$
x_{06}	$(6, 2.93x10^{-7}, 7.87025x10^{-20}, 9.994, 0, 5.22989x10^{-50})$
x_{07}	$(1.04131x10^{-22}, 6.77147, 0.284574, 1.38752x10^{-15}, 0, 19.993)$
x_{08}	$(4.7959x10^{-10}, 2.94553x10^{-18}, 8, 9.992, 0, 2.64283x10^{-185})$
x_{09}	$(0.000195659, 1.71495x10^{-22}, 7.99967, 9.992, 0, 1.17757x10^{-62})$
x_{010}	$(1.12155x10^{-16}, 2.4311x10^{-8}, 8, 5.79344x10^{-9}, 0, 19.992)$
x_{011}	$(3.88564x10^{-10}, 3.73983x10^{-18}, 8, 9.992, 0, 0)$
x_{012}	$(2.61099x10^{-17}, 7., 8.21174x10^{-8}, 5.34553x10^{-19}, 0, 19.993)$
x_{013}	$(0.0288085, 6.96285, 1.12447x10^{-23}, 4.64703x10^{-265}, 14.993, 1.50197x10^{-51})$
x_{014}	$(2.78765x10^{-9}, 3.3229x10^{-19}, 8, 2.23082x10^{-80}, 0, 19.992)$
x_{015}	$(5.98897, 1.03147x10^{-26}, 0.004123, 9.99401, 0, 0)$
x_{016}	$(5.0583x10^{-18}, 6.99998, 0.0000197848, 9.993, 0, 1.61257x10^{-193})$
x_{017}	$(1.19621x10^{-22}, 6.96714, 0.0409594, 9.99299, 0, 1.93423x10^{-211})$
x_{018}	$(8.85815x10^{-20}, 0.0857938, 7.92872, 8.36376x10^{-14}, 0, 19.992)$
x_{019}	$(1.28275x10^{-18}, 8.20286x10^{-8}, 8, 9.992, 0, 3.39803x10^{-166})$
x_{020}	$(3.50275x10^{-20}, 0.0000610008, 7.99995, 6.50151x10^{-59}, 0, 19.992)$

Table 4.5: Trajectories of initial points in a neighborhood of torus go to inside of it at time $t = 100$.

So we can see that for each initial point of the table 4.4 the corresponding trajectory goes to inside of our torus (see table 4.5), and so, we have shown numerically that this torus is an attractor.

Chapter 5

Persistence of the heteroclinic attractor

In chapter 3 we proved that under some conditions the master system (3.3) has a heteroclinic cycle as an attractor and the slave system also has an attractor in the form of a heteroclinic cycle (Figure 3.3) for zero coupling coefficients ($\eta_{ij} = 0$). Thus, the uncoupled system (3.3),(3.4) has a heteroclinic attractor, the direct product of these heteroclinic cycles in the form of a non-smooth two dimensional torus. The goal of this chapter is to show that heteroclinic attractor there exists not only for the uncoupled system but also for master-slave system with a weak coupling.

First, we consider an uncoupled system ($\eta_{ij} = 0$)

$$\begin{cases} \dot{x_1} = x_1(\sigma_1 - x_1 - \rho_{12}x_2 - \rho_{13}x_3) \\ \dot{x_2} = x_2(\sigma_2 - x_2 - \rho_{21}x_1 - \rho_{23}x_3) \\ \dot{x_3} = x_3(\sigma_3 - x_3 - \rho_{31}x_1 - \rho_{32}x_2) \end{cases} \tag{5.1}$$

$$\begin{cases} \dot{y_1} = y_1(\delta_1 - y_1 - \xi_{12}y_2 - \xi_{13}y_3) \\ \dot{y_2} = y_2(\delta_2 - y_2 - \xi_{21}y_1 - \xi_{23}y_3) \\ \dot{y_3} = y_3(\delta_3 - y_3 - \xi_{31}y_1 - \xi_{32}y_2) \end{cases} \tag{5.2}$$

Under conditions formulated in corollary 3.2.1 the system (5.1) has a heteroclinic cycle, say Γ^1 consisting of O_1, O_2, O_3 and $\Gamma_{12}, \Gamma_{23}, \Gamma_{31}$. Under conditions formulated in Proposition 3.2.1, the system (5.2) has a heteroclinic cycle, say Γ^2 consisting of S_1, S_2, S_3 and $\underline{\Gamma_{12}}, \underline{\Gamma_{23}}, \underline{\Gamma_{31}}$. Since Γ^1 and Γ^2 are homeomorphic to circles then it follows that the system (5.1),(5.2) has an invariant set that is cartesian product $\Gamma_0 = \Gamma^1 \text{x} \Gamma^2$, homeomorphic to the two-dimensional torus $\mathbb{T}^2$. The equilibrium points belonging to Γ_0 are:

$$\begin{array}{l} \bar{O_1} = O_1\text{x}S_1,\ \bar{O_2} = O_1\text{x}S_2,\ \bar{O_3} = O_1\text{x}S_3 \\ \dot{O_1} = O_2\text{x}S_1,\ \dot{O_2} = O_2\text{x}S_2,\ \dot{O_3} = O_2\text{x}S_3 \\ \hat{O_1} = O_3\text{x}S_1,\ \hat{O_2} = O_3\text{x}S_2,\ \hat{O_3} = O_3\text{x}S_3. \end{array}$$

The eigenvalues of the system (5.1),(5.2) linearized at these points are presented at the following table

Saddle	Eigenvalues
$\bar{O}_1$	$-\sigma_1, \sigma_2 - \rho_{21}\sigma_1, \sigma_3 - \rho_{31}\sigma_1, -\delta_1, \delta_2 - \xi_{21}\delta_1, \delta_3 - \xi_{31}\delta_1$
$\bar{O}_2$	$-\sigma_1, \sigma_2 - \rho_{21}\sigma_1, \sigma_3 - \rho_{31}\sigma_1, -\delta_2, \delta_3 - \xi_{32}\delta_2, \delta_1 - \xi_{12}\delta_2$
$\bar{O}_3$	$-\sigma_1, \sigma_2 - \rho_{21}\sigma_1, \sigma_3 - \rho_{31}\sigma_1, -\delta_3, \delta_1 - \xi_{13}\delta_3, \delta_2 - \xi_{23}\delta_3$
$\dot{O}_1$	$-\sigma_2, \sigma_3 - \rho_{32}\sigma_2, \sigma_1 - \rho_{12}\sigma_2, -\delta_1, \delta_2 - \xi_{21}\delta_1, \delta_3 - \xi_{31}\delta_1$
$\dot{O}_2$	$-\sigma_2, \sigma_3 - \rho_{32}\sigma_2, \sigma_1 - \rho_{12}\sigma_2, -\delta_2, \delta_3 - \xi_{32}\delta_2, \delta_1 - \xi_{12}\delta_2$
$\dot{O}_3$	$-\sigma_2, \sigma_3 - \rho_{32}\sigma_2, \sigma_1 - \rho_{12}\sigma_2, -\delta_3, \delta_1 - \xi_{13}\delta_3, \delta_2 - \xi_{23}\delta_3$
$\hat{O}_1$	$-\sigma_3, \sigma_1 - \rho_{13}\sigma_3, \sigma_2 - \rho_{23}\sigma_3, -\delta_1, \delta_2 - \xi_{21}\delta_1, \delta_3 - \xi_{31}\delta_1$
$\hat{O}_2$	$-\sigma_3, \sigma_1 - \rho_{13}\sigma_3, \sigma_2 - \rho_{23}\sigma_3, -\delta_2, \delta_3 - \xi_{32}\delta_2, \delta_1 - \xi_{12}\delta_2$
$\hat{O}_3$	$-\sigma_3, \sigma_1 - \rho_{13}\sigma_3, \sigma_2 - \rho_{23}\sigma_3, -\delta_3, \delta_1 - \xi_{13}\delta_3, \delta_2 - \xi_{23}\delta_3$

Table 5.1: Eigenvalues of every equilibrium point of the heteroclinic network.

It follows from the conditions imposed in Section 3.2. that each of these points has two-dimensional unstable and four-dimensional stable manifolds. Denote by $H(A \to B)$ a heteroclinic trajectory joining the equilibrium points A and B. Then,

$$\Gamma_{12} = H(O_1 \to O_2), \Gamma_{23} = H(O_2 \to O_3), \Gamma_{31} = H(O_3 \to O_1),$$
$$\underline{\Gamma_{12}} = H(S_1 \to S_2), \underline{\Gamma_{23}} = H(S_2 \to S_3), \underline{\Gamma_{31}} = H(S_3 \to S_1)$$

and we have

$$\begin{aligned}
&H(\bar{O}_1 \to \bar{O}_2) = O_1 \text{ x } \underline{\Gamma_{12}},\ H(\bar{O}_2 \to \bar{O}_3) = O_1 \text{ x } \underline{\Gamma_{23}},\ H(\bar{O}_3 \to \bar{O}_1) = O_1 \text{ x } \underline{\Gamma_{31}},\\
&H(\dot{O}_1 \to \dot{O}_2) = O_2 \text{ x } \underline{\Gamma_{12}},\ H(\dot{O}_2 \to \dot{O}_3) = O_2 \text{ x } \underline{\Gamma_{23}},\ H(\dot{O}_3 \to \dot{O}_1) = O_2 \text{ x } \underline{\Gamma_{31}},\\
&H(\hat{O}_1 \to \hat{O}_2) = O_3 \text{ x } \underline{\Gamma_{12}},\ H(\hat{O}_2 \to \hat{O}_3) = O_3 \text{ x } \underline{\Gamma_{23}},\ H(\hat{O}_3 \to \hat{O}_1) = O_3 \text{ x } \underline{\Gamma_{31}},\\
&H(\bar{O}_1 \to \dot{O}_1) = \Gamma_{12} \text{ x } S_1,\ H(\dot{O}_1 \to \hat{O}_1) = \Gamma_{23} \text{ x } S_1,\ H(\hat{O}_1 \to \bar{O}_1) = \Gamma_{31} \text{ x } S_1,\\
&H(\bar{O}_2 \to \dot{O}_2) = \Gamma_{12} \text{ x } S_2,\ H(\dot{O}_2 \to \hat{O}_2) = \Gamma_{23} \text{ x } S_2,\ H(\hat{O}_2 \to \bar{O}_2) = \Gamma_{31} \text{ x } S_2,\\
&H(\bar{O}_3 \to \dot{O}_3) = \Gamma_{12} \text{ x } S_3,\ H(\dot{O}_3 \to \hat{O}_3) = \Gamma_{23} \text{ x } S_3,\ H(\hat{O}_3 \to \bar{O}_3) = \Gamma_{31} \text{ x } S_3.
\end{aligned}$$

The surface Γ_0 consists of 9 "rectangles"(see Fig 3.7). Let us consider any of them, for instance, the one boundary of which consists of the heteroclinic trajectories: $H(\bar{O}_1 \to \bar{O}_2)$, $H(\bar{O}_2 \to \dot{O}_2)$, $H(\bar{O}_1 \to \dot{O}_1)$, $H(\dot{O}_1 \to \dot{O}_2)$. Denote it by π_0. Since all points $\bar{O}_1, \bar{O}_2, \dot{O}_1, \dot{O}_2$ belong to the 4-dimensional invariant subspace $x_3 = y_3 = 0$, all heteroclinic trajectories in the boundary of π_0 also belong to this subspace, and each point in $Int(\pi_0)$ is a direct product of two points one of which belong to Γ_{12} and another to $\underline{\Gamma_{12}}$, i.e, it belongs to the same subspace, then $\pi_0 \subset \{x_3 = y_3 = 0\} := {\mathbb{R}_0}^4$. Inside ${\mathbb{R}_0}^4$ we have

$$dimW^u(\bar{O}_1) = 2, dimW^u(\bar{O}_2) = dimW^u(\dot{O}_1) = 1, dimW^u(\dot{O}_2) = 0.$$

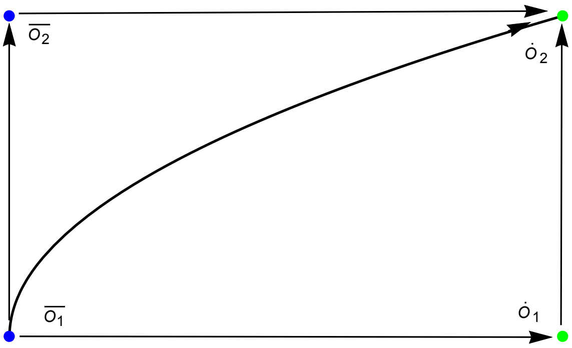

Figure 5.1: The rectangule π_0.

Lemma 5.0.1 *The trajectory $H(\bar{O}_1 \to \bar{O}_2)$ belongs to the transversal in ${\mathbb{R}_0}^4$ intersection $W^u(\bar{O}_1) \cap W^s(\bar{O}_2)$.*

Proof. The trajectory $H(\bar{O}_1 \to \bar{O}_2) = O_1 \text{ x } \underline{\Gamma_{12}}$ belongs to the plane $x_2 = 0, x_1 = \sigma_1$. The local stable manifold of the point $\bar{O}_2$ is a piece of the plane $x_2 = 0$, so, the vector $\overrightarrow{V} = (0, 1, 0, 0)$ is perpendicular to this plane. The cartesian product Γ_{12} x $\underline{\Gamma_{12}}$ is a part of $W^u(\bar{O}_1)$, so at a point $p \in H(\bar{O}_1 \to \bar{O}_2)$, close to $\bar{O}_1$, there is a tangent vector to Γ_{12} x $\underline{\Gamma_{12}}$ that has the form $\overrightarrow{W} = (\alpha, \beta, \gamma, \delta)$ where the vector (α, β) is close to the eigenvector $(\frac{\rho_{12}\sigma_1}{-\sigma_1+\rho_{21}\sigma_1-\sigma_2}, 1)$ on the plane $y_1 = y_2 = 0$ corresponding to the eigenvalue $\sigma_2 - \rho_{21}\sigma_1 > 0$. Thus, $\beta \neq 0$ if a point p is close enough to $\bar{O}_1$ (and the projection of p onto the (x_1, x_2)-plane is close enough to O_1). Since the scalar product $(\overrightarrow{V}, \overrightarrow{W}) = \beta \neq 0$, it follows that the intersection $W^u(\bar{O}_1) \cap W^s(\bar{O}_2)$ is transversal at the point p and, consequently, it is transversal at every point of $H(\bar{O}_1 \to \bar{O}_2)$. ■

Lemma 5.0.2 *The trajectory $H(\bar{O}_1 \to \dot{O}_1)$ belongs to the transversal in ${\mathbb{R}_0}^4$ intersection $W^u(\bar{O}_1) \cap W^s(\dot{O}_1)$.*

Proof is similar to that for Lemma 5.0.1.

Theorem 5.0.1 *If saddle values of all saddle equilibrium points belonging to Γ_0 are greater than 1, then Γ_0 is an attractor.*

Proof. Because of the construction of Γ_0, we can start with an initial point p_0 that is close to one of the equilibrium points in Γ_0. For the sake of definiteness let it will be $\bar{O}_1$. Let $\epsilon = dist(p_0, W^s(\bar{O}_1))$. It follows (see the book [5]) that the representative point on the trajectory going through p_0 leaves a neigborhood of $\bar{O}_1$ at a point p_1 such that $dist(p_1, W^u(\bar{O}_1)) < \epsilon^\nu$, where $\nu > 1$ and can be estimated by the saddle value of the point O_1. Then the trajectory going through p_1 follows a trajectory on the $W^u(\bar{O}_1)$ and comes eventually, after a finite time to a point p_2 in a small neigborhood of the next equilibrium point, say $\tilde{O}$ ($\tilde{O}$ is either $\bar{O}_2$ or $\dot{O}_1$ or $\dot{O}_2$.), so that $dist(p_2, W^s(\tilde{O})) \leq c\epsilon^\nu$, where $c = constant$. If ϵ is small enough we have $c\epsilon^\nu < \frac{1}{2}\epsilon$. Then we repeat the previuos consideration for the point $\tilde{O}$ replacing ϵ to $\frac{1}{2}\epsilon$. We do it again and again to see that $dist(p_{2k+1}, \Gamma_0) < \frac{1}{2^k}\epsilon$, where p_{2k+1} is the representative point on the trajectory after the time the trajectory intersects k successive neigborhoods. Let us remark that we should choose the smallness of ϵ only finitely many times, since Γ_0 consists of finitely many rectangles. Thus, the trajectory goes to Γ_0 as $t \to \infty$. ■

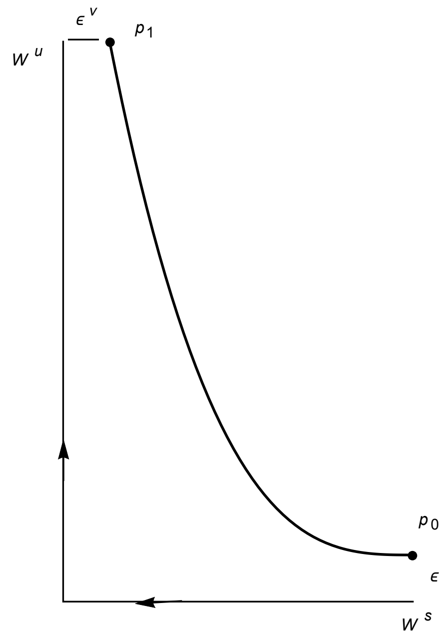

Figure 5.2: Passing the neigborhood of a saddle point.

Now we consider the system (3.3),(3.4) for small values of the parameters η_{ij}.

Theorem 5.0.2 *There is an $\epsilon > 0$ such that if $|\eta_{ij}| < \epsilon$, then the system (3.3),(3.4) has an invariant two-dimensional surface Γ_η homeomorphic to Γ_0 and close to it.*

Proof. As we show bellow such a surface will consist of "rectangles" bounded by the heteroclinic trajectories joining saddle equilibrium points. For the sake of definiteness, we consider the rectangle π_0 and prove the existence of a rectangle π_2 close to π_0. We show its existence in ${\mathbb{R}_0}^4$.

(*i*) For $\eta_{ij} = 0$ the heteroclinic trajectories $H(\bar{O}_1 \to \bar{O}_2)$, $H(\bar{O}_2 \to \dot{O}_2)$, $H(\bar{O}_1 \to \dot{O}_1)$, $H(\dot{O}_1 \to \dot{O}_2)$ belong to transversal intersections of stable and unstable manifolds (Lemmas 5.0.1 and 5.0.2), so for small values of η_{ij} They are still exist and belong to the transversal intersections of stable and unstable manifolds of saddle equilibrium points.

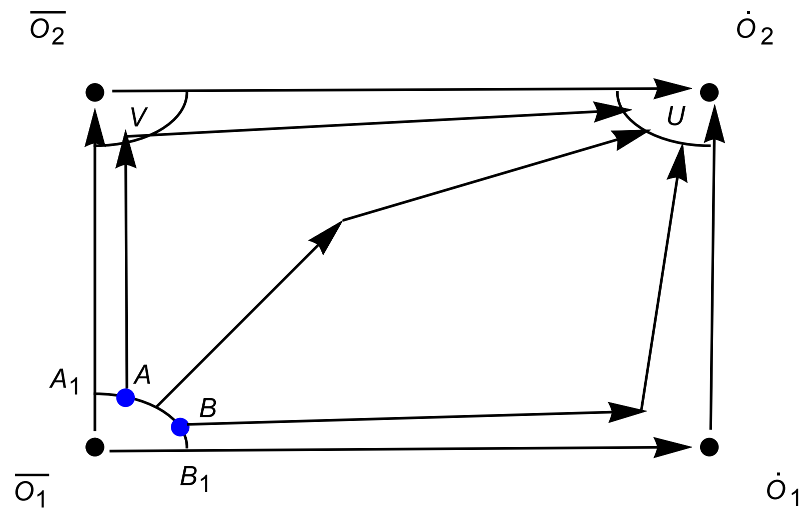

Figure 5.3: Trajectories on π_η.

(ii) The point $\dot{O}_2$ is the stable node $\mathbb{R}_0{}^4$ for $\eta_{ij} = 0$, so, for small values of η_{ij} it is still the stable node, and there exists an absorbing region U with the maximal attractor $\dot{O}_2$ inside, see Figure 5.3.

(iii) The local unstable manifold of $\bar{O}_1$, say $W^u(\eta)$ depends smoothly on parameters, so, for small values of η_{ij} it is $C' \equiv C^{cst}$ close to the $W^u(0)$, the local unstable manifold for $\eta_{ij} = 0$. Therefore if we choose an initial point p on the interval (A, B) on $W^u(\eta)$, see Figure 5.3, it will be close to a point $p_0 \in W^u(0)$. For the uncoupled system ($\eta_{ij} = 0$) the trajectory going to through p_0 reaches U in finite time that can be estimated from above by a constant depending only on the $dist(A, A_1)$, $dist(B, B_1)$. Therefore the trajectory of the system (3.3),(3.4) going through p also comes to U in finite time, if values η_{ij} are small enough, because of theorem of continuous dependence of solutions of ODE on parameters.

(iv) It is remain to show that the representative point in a trajectory going through a point $p \in (A, A_1)$ or $p \in (B, B_1)$ eventually comes to U. Assume that A is so close to A_1 that the trajectory going through it intersects a small neigborhood V of $\bar{O}_2$ at a point p_0, such that $dist(p_0, W^s(\bar{O}_2)) < \delta$. We apply results of the book [5] to show that $dist(p_1, W^u(\bar{O}_2)) < \delta^\nu$, (see Figure 5.2) where p_1 is a point on the trajectory at the instant when it leaves the neighborhood V. It means that if δ is small enough, then the point p_1 is close to a point on the heteroclinic trajectory $H(\bar{O}_2 \to \dot{O}_2)$, and therefore the trajectory going through p_1 comes to U in finite time.

(v) The proof for the points on (B_1, B) is the same. Thus, we proved that every trajectory going through points on $W^u(\eta) \cap (A_1, B_1)$ comes to U and, then, goes to $\dot{O}_2$. The points on these trajectories together with saddle equilibrium points form a desired rectangle π_η. It follows from the construction that $\pi_\eta \subset \mathbb{R}_0{}^4$. The union of these rectangles form the surface Γ_η. The fact that Γ_η is homeomorphic to Γ_0 follows directly from the proof. ■

Theorem 5.0.3 *If all saddles belonging to* Γ_η *has saddle values greater than 1, then* Γ_η *is an attractor.*

The proof is the same as that for theorem 5.0.1.

The attractor Γ_η is evidently non chaotic, it consists of several equilibrium points and heteroclinic trajectories join them. But there is a logical possibility that trajectories in its basin are Lyapunov unstable. The possibility could be checked numerically.

Chapter 6

Conclusions and future work

In the thesis we studied dynamics of two Lotka-Volterra systems coupled in the master-slave way. We have obtained the following results:

1) We found conditions in the form of inequalities on the coefficients, under which the system has a heteroclinic network consisting of saddle equilibrium points (with 2-dimensional unstable manifolds) and joinig them heteroclinic trajectories.

2) We have shown numerically that this heteroclinic network can serve as a skeleton of a two-dimensional non-smooth torus that is an attractor.

3) We proved analytically that such torus exists when coefficients of coupling are small enough.

In the future we plan to study numerically dynamics of this system in a beside of this torus. We hypothesize that the system may manifest instability of transient trajectories in the form of so called weak transient chaos. We plan calculate special Lyapunov exponents and expect that they will be positive. We plan to write an article on the base of results of the thesis.

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