UNIVERSIDAD AUTÓNOMA DE SAN LUIS POTOSÍ

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CLASSIFYING CELLULAR AUTOMATA USING ORDER AND CHAOS

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PRESENTA:

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DIRECTOR DE TESIS:

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ALUMNO:

LUGUIS DE LOS SANTOS BAÑOS

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DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS

Yo, Luguis de los Santos Baños, estudiante del Posgrado en Ciencias Aplicadas de la Facultad de Ciencias de la Universidad Autónoma de San Luis Potosí, como autor de la tesis "CLASSIFYING CELLULAR AUTOMATA USING ORDER AND CHAOS", declaro que la tesis es una obra original, inédita, auténtica, personal, que se han citado las fuentes correspondientes y que en su ejecución se respetaron las disposiciones legales vigentes que protegen los derechos de autor y de propiedad intelectual e industrial. Las ideas, doctrinas, resultados y conclusiones a los que he llegado son de mi absoluta responsabilidad.

Resumen

Probaremos que un autómata celular en un subshift transitivo es casi equicontinuo o sensible. Por otro lado, construimos un autómata celular en un full shift que no es ni casi equicontinuo en promedio ni sensible en promedio. Además, utilizando algún tipo de skew product "local" entre un shift y un autómata celular que parece un odómetro, mostramos que existe un autómata celular casi diam-mean equicontinuo pero no casi equicontinuo.

Abstract

We show that a cellular automaton on a transitive subshift is either almost equicontinuous or sensitive. On the other hand, we construct a cellular automaton on a full subshift that is neither almost mean equicontinuous nor mean sensitive. Furthermore, using some type of "local" skew product between a shift and an odometer looking cellular automaton, we show that there exists an almost diam-mean equicontinuous cellular automaton that is not almost equicontinuous.

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Introduction

Sensitivity to initial conditions (or simply sensitivity) is one of the classical notions of chaos on dynamical systems. Such notion was introduced for topological dynamical systems by Guckenheimer in [17]. By a **topological dynamical system (shortly TDS)** we mean a pair (X, T) such that X is a compact metric space (with metric d) and $T : X \to X$ is a continuous map. A TDS is **sensitive** if given any region of X, there exist two points in the region and a time unit $n \in \mathbb{N}$ such that the n^{th} iterates of the two points under the map T are "significantly separated"; that is: if there exists $\varepsilon > 0$ such that for every non-empty open set $U \subseteq X$ there exist $x, y \in U$ and $n > 0$ such that $d(T^n x, T^n y) > \varepsilon$. A notion of order that contrasts sensitivity is equicontinuity (or Lyapunov stability); a TDS is **equicontinuous** if $\{T^n\}_{n\in\mathbb{N}}$ is an equicontinuous family of maps. One may also study this notion locally. A point $x \in X$ is an **equicontinuity point** of (X, T) if the diameter of the images of a small ball around x will always stay small; that is: if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{diam}(T^i B_\delta(x)) < \varepsilon$ for every $i \in \mathbb{N}$. We say a TDS is **almost equicontinuous** if the equicontinuity points are dense in X. Using sensitivity and equicontinuity one can classify transitive topological dynamical systems (see Definition 1.3). Akin, Auslander and Berg proved that any transitive TDS is either sensitive or almost equicontinuous [1] (a generalization of the Auslander-Yorke dichotomy in [2], which is only for minimal sets).

The dichotomy mentioned above has some limitations, since sensitivity is not a very strong form of chaos. The "largeness" of the set of all $n \in \mathbb{N}$ where this "significant separation" happens for a particular pair of points in a given region, can be though of as a measure of how sensitive the system is. If the set turns out to be rather "thin" with large gaps between consecutive entries, then we may have some excuse for treating the system as practically non-sensitive. So, if the large gaps between consecutive entries do not matter, we can take the averages of the distances of the first n^{th}iterations, this generates a sequence. If this sequence does not converge, we could say that the TDS is **mean sensitive** (defined by García-Ramos [13]); that is: if there exists $\varepsilon > 0$ such that for every non-empty open set $U \subseteq X$ there exist $x, y \in U$ such that

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} > \varepsilon.$$

In contrast, we have that a point $x \in X$ is a **mean equicontinuity point** (defined by Fomin [9]) if the diameter of the images of a small ball around x will stay small on average; that is: for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} \leq \varepsilon.$$

Similar to the classic case, one can classify transitive TDSs using the mean notions; that is: a transitive TDS is either mean sensitive or almost mean equicontinuous [22, 13]. Mean equicontinuity/sensitivity has been studied in recent papers (see for instance [8, 14, 18, 15, 10] or the survey [23]). The set of mean sensitive TDSs is properly contained in the set of sensitive TDSs.

A weaker notion of equicontinuity but stronger than mean equicontinuity, **diam-mean equicontinuity**, requires that the diameter of small balls to stay small on average (see Definition 1.15). The notion of diam-mean equicontinuity has been used to characterize regularity properties of the maximal equicontinuous factor [14], which are natural in the context of aperiodic order and mean equicontinuity (introduced in [9, 26]).

Cellular automata (CA) were introduced by Ulam and von Neumann to model the evolution of cells. We say that (X, T) is a **cellular automaton (CA)** if X is a subshift and $T : X \to X$ is continuous and commutes with σ, i.e., $\sigma \circ T = T \circ \sigma$. Cellular automata can be studied in the context of TDS. For cellular automata, equicontinuity is strongly connected to local periodicity; that is: if $\{T^n x_i\}_{n\in\mathbb{N}}$ is periodic for all $i \in \mathbb{Z}$ [11]. Given a CA, it is not difficult to check that a point is equicontinuous if and only if it is locally eventually periodic; that is: if $\{T^n x_i\}_{n\in\mathbb{N}}$ is eventually periodic for all $i \in \mathbb{Z}$ (in Proposition 2.12). However, for the mean versions of equicontinuity this is not true. The notion of sensitivity in CA has been studied in many papers (see for example, [20, 24, 16, 3, 27, 11]).

Kurka proved that any CA (not necessarily transitive) is either sensitive or almost equicontinuous [21]. So, it is natural to seek the answers of the following questions:

- Is the set of almost equicontinuous CA properly contained in the set of almost diam-mean equicontinuous CA?
- Is the set of almost diam-mean equicontinuous CA properly contained in the set of almost mean equicontinuous CA?
- Can Kurka's dichotomy be adapted for the (diam-)mean versions of equicontinuity/sensitivity?
- Does there exist an almost mean equicontinuous CA such that is diam-mean sensitive?

In Chapter 1 we will study some basic dynamical notions, such as transitivity, which plays an important role in the dichotomy of these systems.

In Chapter 2, we introduce cellular automata (CA). We use Theorem 2.2 to give an equivalent definition of CA. We will see that a CA is almost equicontinuous if and only if is not sensitive (see Proposition 2.8). Transivity does not play a role in this dichotomy, at least not directly. In Proposition 2.12 will show how equicontinuity and periodicity are related locally.

In Chapter 3, we provide four examples (the main results of the thesis). The first two examples (The Pacman and The Pacman level 2 CA) are the first examples in the study of mean equicontinuity/sensitivity on CA. In Section 3.1, to proof Theorem 3.14, we construct an almost mean equicontinuous CA (Pacman) that is not almost equicontinuous. In Section 3.2, to proof Theorem 3.18, we construct a CA (Pacman level 2) that is neither mean sensitive nor almost mean equicontinuous. So, Kurka's dichotomy does not hold for the mean notions on cellular automata. In Section 3.3, to proof Theorem 3.40, we take some form of local skew-product between a very regular CA (similar to an odometer) and a very chaotic CA on the shift map to construct a CA that is almost diam-mean equicontinuous but not almost equicontinuous. In

Section 3.4, we construct a CA that is neither almost diam-mean equicontinuous nor diam-mean sensitive (in Theorem 3.43). In Section 3.5, we show that the Pacman and the Pacman level 2 CA are cofinitely sensitive. It is easy to see that every cofinitely sensitive TDS is diam-mean sensitive. So, there exist CA that are almost mean equicontinuous and diam-mean sensitive, and diam-mean sensitive CA such that they are neither almost mean equicontinuous nor mean sensitive.

In Chapter 4, we give a set of questions on various topics related directly or indirectly to the results of this thesis.

Chapter 1

Topological dynamical systems

The main abstract object of study in this theses are topological dynamical systems.

Definition 1.1. A **topological dynamical system (TDS)** is a pair (X, T) where X is a compact metric space (with metric d) and $T : X \to X$ is continuous map.

Definition 1.2. Let (X, T) be a TDS.

1. A point $x \in X$ is **periodic**, if there exists $p > 0$ with $T^p x = x$. The least p with this property is called the **period** of x.

2. A point $x \in X$ is **eventually periodic**, if $T^m x$ is periodic for some $m \geq 0$.

Definition 1.3. Let (X, T) be a TDS. We say that (X, T) is **transitive** if for every pair of non-empty open sets U and V there exists $n > 0$ such that $T^{-n}U \cap V \neq \emptyset$. A **transitive point** is a point such that its orbit is dense.

For example a TDS that consists of one periodic orbit is transitive, but a TDS formed by two disjoint periodic orbits is not transitive.

A subset of a topological space is **residual (or comeagre)** if it includes the intersection of countably many dense open sets. By the Baire category theorem, residual sets of complete metrizable spaces are always dense.

Definition 1.4. Let (X, T) be a TDS and $x \in X$.

1. The point x is an **equicontinuity point** of (X, T) if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \ \forall y \in B_\delta(x),\ \forall n \geq 0,\ d(T^n x, T^n y) < \varepsilon.$$

   The set of equicontinuity points of (X, T) is denoted by EQ.

2. (X, T) is **equicontinuous** if $EQ = X$.

3. (X, T) is **almost equicontinuous** if EQ is a residual set.

4. (X, T) is **sensitive** if there exists $\varepsilon > 0$ such that for every non-empty open set $U \subseteq X$ there exist $x, y \in U$ and $n \neq 0$ such that

$$d(T^n x, T^n y) > \varepsilon.$$

Clearly, if (X, T) is equicontinuous, then it is almost equicontinuous. The other way around is not true (see Example 2.7). Straightforward examples of equicontinuous TDS are the constant function or any finite periodic system. An easy sensitive example is the full shift, for this we need to define symbolic dynamical systems.

Definition 1.5.

1. Given a finite non-singular set A (called an alphabet), we define the A-**full shift** as $A^{\mathbb{Z}}$. If X is the A-full shift for some finite A we say that X is a **full shift**.

2. Given $x \in A^{\mathbb{Z}}$, we represent the i-th coordinate of x as x_i. Also, given $i, j \in \mathbb{Z}$ with $i < j$, we define the finite word $x_{[i,j]} = x_i \dots x_j$. We denote by A^n the set of words of A of length $n \geq 1$.

3. We endow any full shift with the metric

$$d(x, y) \quad = \quad \begin{cases} 2^{-i} & \text{if } x \neq y \quad \text{where } i = \min\{|j| : x_j \neq y_j\}; \\ 0 & \text{otherwise.} \end{cases}$$

   This metric generates the same topology as the prodiscrete topology.

4. For any full shift $A^{\mathbb{Z}}$, we define the shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by $\sigma(x)_i = x_{i+1}$. The shift map is continuous (with respect to the previously defined metric).

5. We say X is a **subshift** (or shift space) if $X \subseteq A^{\mathbb{Z}}$ is closed and σ-invariant ($\sigma(X) = X$).

Let $n \in \mathbb{N}$. We will denote the **balls of radius** 2^{-n} with $B_n(x)$. That is,

$$B_n(x) = \{y \in A^{\mathbb{Z}} : x_i = y_i \forall i \in [-n, n]\}.$$

A **cylinder** in a product space $A^{\mathbb{Z}}$ is any set

$$[w] = \{x \in A^{\mathbb{Z}} ; x_{[0,|w|)} = w\},$$

where $w \in A^n$ and $n > 0$. A cylinder set is a ball. If $x \in [w]$ then $[w] = B_{|w|-1}(x)$, so $[w]$ is an open set. The complement of $[w]$ is a finite union of cylinders, so it is open too. Thus every cylinder is **clopen** (close and open). Consider a nonempty open set $U \subseteq A^{\mathbb{Z}}$. For any $x \in U$ there exists $n \geq 0$ such that $B_n(x) \subseteq U$. Thus, there exists $w \in A^{n+1}$ such that $[w] = B_n(x)$.

It is not difficult to check that the full shift is sensitive, in fact every subshift without isolated points is sensitive. An example of an equicontinuous but not periodic TDS is the **odometer**. This TDS is called odometer because the dynamics on finite sections look exactly like the mileage meter on a car. The following example defines it in detail.

Example 1.6. Let $X = \mathbb{Z}_2^{\mathbb{N}}$ equipped with the prodiscrete topology. We also give X a group structure with the operation of adding each coordinate with carrying. That is, if $y = x \cdot x'$, then $y_i = x_i + x'_i \text{mod} 2$ unless $x_{i-1} + x'_{i-1} > 0$ and $y_{i-1} = 0$, in this case $y_i = x_i + x'_i + 1 \text{mod} 2$. The **binary odometer** is the TDS (X, T) where $Tx = x \cdot (1, 0, 0, ...)$ (for a survey on odometers see [7]). If $x_i = x'_i$ for all $i \in [0, n]$ then $Tx_i = Tx'_i$ for all $i \in [0, n]$. This implies that (X, T) is equicontinuous. It is not difficult to check that this TDS has no periodic points.

Sensitivity and almost equicontinuity can be used to give a classification of transitive topological dynamical systems.

Theorem 1.7 (Akin-Auslander-Berg dichotomy [1])**.** *Transitive topological dynamical systems are sensitive if and only if they are not almost equicontinuous.*

Proof. Let (X,T) be a transitive system and $\varepsilon > 0$. Firstly, we are going to show that the set

$$EQ_\varepsilon := \{x \in X : \exists \delta, \forall y, z \in B_\delta(x), \forall n \geq 0, d(T^n y, T^n z) < \varepsilon\}$$

is open. Let $x \in EQ_\varepsilon$. Therefore, there exists a $\delta > 0$ such that for every $y, z \in B_\delta(x)$ satisfies that $d(T^i y, T^i z) < \varepsilon$ for all $n > 0$. Let $y \in B_\delta(x)$. Let us take $\delta' < \min\{d(x,y), \delta - d(x,y)\}$. Then, $B_{\delta'}(y) \subseteq B_\delta(x)$. Hence, $B_\delta(x) \subseteq EQ_\varepsilon$. Therefore, EQ_ε is open.

Assume that EQ_ε is nonempty and non dense. Then $U = X \setminus \overline{EQ_\varepsilon}$ is open and nonempty. So, since (X,T) is transitive there exists $n > 0$ such that

$$\emptyset \neq U \cap T^{-n} EQ_\varepsilon \subseteq U \cap EQ_\varepsilon = \emptyset,$$

and this is a contradiction. Thus, EQ_ε for any $\varepsilon > 0$, is either empty or dense. If for all $\varepsilon > 0$, EQ_ε is nonempty. Then, $EQ = \bigcap_{m \geq 1} EQ_{\frac{1}{m}}$ is a residual set. So, (X,T) is almost equicontinuous. If $EQ_\varepsilon = \emptyset$ for some $\varepsilon > 0$, then the system is sensitive: For any $x \in X$ and for any $\delta > 0$ there exist $y, z \in B_\delta(x)$ and $n \geq 0$ such that $d(T^n y, T^n z) \geq \varepsilon$. It follows that either $d(T^n y, T^n x) \geq \frac{\varepsilon}{2}$ or $d(T^n z, T^n x) \geq \frac{\varepsilon}{2}$.

□

Without transitivity we may have cases just as the following example.

Example 1.8. Let

$$X = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 0\} \cup \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + z^2 = 1, y = 0\},$$

and

$$T(r\cos t, r\sin t, 0) = (r\cos 2t, r\sin 2t, 0)$$
$$T(1 - \cos t, 0, \sin t) = (1 - \cos 2t, 0, \sin 2t).$$

The TDS (X,T) does not have equicontinuity points but is not sensitive.

It is easy to check that every isometry is equicontinuous. In particular, irrational rotations of the circle are also equicontinuous non-periodic TDS. If we code this system into a subshift we obtain a very regular TDS with no equicontinuity points. This means that equicontinuity is a very strong property.

Example 1.9. Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and consider the rigid translation $R_\alpha : [0,1) \to [0,1)$ given by $R_\alpha(t) = t + \alpha (\bmod 1)$, for $t \in [0,1)$. Consider the partition $\{[0, 1-\alpha), [1-\alpha, 1)\}$ of the unit interval and define a coding map $I_\alpha : [0,1) \to \{0,1\}$ by

$$I_\alpha(t) = \begin{cases} 0 & \text{if} \quad t \in [0, 1-\alpha), \\ 1 & \text{if} \quad t \in [1-\alpha, 1), \end{cases}$$

for $t \in [0,1)$. The one-sided **Sturmian subshift** with parameter α is the space

$$X_\alpha := \overline{\{\{I_\alpha(R_\alpha^i(t))\}_{i \in \mathbb{N}} : t \in [0,1)\}} \subseteq \{0,1\}^{\mathbb{N}},$$

together with the shift map $\sigma_\alpha : X_\alpha \to X_\alpha$ given by $\sigma_\alpha = \sigma|_{X_\alpha}$, for $x = (x_i)_{i \in \mathbb{N}} \in X_\alpha$ and for $i \in \mathbb{N}$. It is easy to see that $(X_\alpha, \sigma_\alpha)$ is sensitive.

We now introduce the mean versions of equicontinuity and sensitivity. Mean equicontinuity is weaker than equicontinuity and sensitivity is weaker than mean sensitivity.

Definition 1.10. Let (X,T) be a TDS and $x \in X$.

1. We call x a **mean equicontinuity point** of (X,T) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x,y) < \delta$, then

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} \leq \varepsilon.$$

   We denote the set of mean equicontinuty points of (X,T) by EQ^M.

2. The TDS (X,T) is **mean equicontinuous** if $X = EQ^M$.

3. The TDS (X,T) is **almost mean equicontinuous** if EQ^M is a residual set.

4. The TDS (X,T) is **mean sensitive** if there exists $\varepsilon > 0$ such that for every non-empty open set $U \subseteq X$ there exist $x, y \in U$ such that

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} > \varepsilon.$$

The Sturmian subshift is mean equicontinuous but we will not prove this here. An easier example is the following.

Example 1.11. Let $X \subset \{0,1\}^{\mathbb{Z}}$ be the subshift consisting of sequences that contain at most one 1. Notice that, for all $x, y \in X$ we have that

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(\sigma^i x, \sigma^i y)}{n+1} = 0.$$

So, we have that (X,σ) is mean equicontinuous. Furthermore, one can easily check that 0^∞ is not an equicontinuity point.

Proposition 1.12. *[13, Lemma 5]*
Let (X,T) be a TDS and $\varepsilon > 0$. Set

$$EQ_\varepsilon^M = \{x \in X : \exists\ \delta > 0,\ \forall\ y, z \in B_\delta(x),\ \limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i y, T^i z)}{n+1} < \varepsilon\}. \tag{1.1}$$

Then, EQ_ε^M is open and $EQ^M = \bigcap_{m>0} EQ_{\frac{1}{m}}^M$. Furthermore, if EQ^M is nonempty, then EQ^M is dense if and only if it is a residual set.

Proof. Let $x \in EQ_\varepsilon^M$. Therefore, there exists $\delta > 0$ such that for every $y, z \in B_\delta(x)$ it follows that:

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i y, T^i z)}{n+1} < \varepsilon.$$

Let $y \in B_\delta(x)$. Let us take $\delta' < \min\{d(x,y), \delta - d(x,y)\}$. Then, $B_{\delta'}(y) \subseteq B_\delta(x)$. Hence, $B_\delta(x) \subseteq EQ_\varepsilon^M$ and EQ_ε^M is open.

Next, we are going to proof that $EQ^M = \bigcap_{m>0} EQ_{\frac{1}{m}}^M$.

$\subseteq$: Let $x \in EQ^M$ and $\varepsilon > 0$. By hypothesis, there exists $\delta > 0$ such that for every $p \in B_\delta(x)$ we have that:

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i x, T^i p)}{n+1} < \frac{\varepsilon}{2}.$$

Now, let $y, z \in B_\delta(x)$. We have that:

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i y, T^i z)}{n+1} \\
\leq\ & \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i y, T^i x) + \sum_{i=0}^n d(T^i x, T^i z)}{n+1} \\
\leq\ & \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i x, T^i y)}{n+1} + \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i x, T^i z)}{n+1} \\
<\ & \varepsilon.
\end{aligned}$$

Therefore $x \in \bigcap_{m>0} EQ^M_{\frac{1}{m}}$.

$\supseteq$: This part is straightforward to show.

By the Baire theorem (see [21]), we have that EQ^M is dense if and only if it is a residual set. $\square$

Lemma 1.13. *[22, Proposition 4.2]*
Let (X,T) be a topological dynamical system. The sets EQ^M and EQ^M_ε are inversely invariant, i.e., $T^{-n}EQ^M \subseteq EQ^M$, $T^{-n}EQ^M_\varepsilon \subseteq EQ^M_\varepsilon$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. Let $x \in X$ with $Tx \in EQ^M_\varepsilon$. Choose $\delta > 0$ satisfying (1.1) from the definition of EQ^M_ε for Tx. By the continuity of T, there exists $\eta > 0$ such that $d(Ty, Tz) < \delta$ for any $y, z \in B_\eta(x)$. Thus,

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n d(T^i y, T^i z) = \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n d(T^i(Ty), T^i(Tz)) < \varepsilon.$$

This implies that $x \in EQ^M_\varepsilon$.

Observe that, for any $\varepsilon' > \varepsilon > 0$ we have that $EQ^M_\varepsilon \subseteq EQ^M_{\varepsilon'}$. Hence, $T^{-n}EQ^M_\varepsilon \subseteq T^{-n}EQ^M_{\varepsilon'}$ for all $n \in \mathbb{Z}_{\geq 0}$. So, we have that

$$T^{-n}EQ^M = \bigcap_{m\geq 0} T^{-n}EQ^M_{2^{-m}} \subseteq \bigcap_{m\geq 0} EQ^M_{2^{-m}} = EQ^M.$$

$\square$

The Akin-Auslander-Berg dichotomy can also be stated for the mean versions of equicontinuity/sensitivity.

Theorem 1.14 (Mean Akin-Auslander-Berg dichotomy [22, 13])**.** *Transitive topological dynamical systems are mean sensitive if only if they are not almost mean equicontinuous.*

Proof. Let $x \in X$ be a transitive point such that there exists $\varepsilon > 0$ such that for every $\delta > 0$ there is $y \in B_\delta(x)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n d(T^i x, T^i y) > \varepsilon. \tag{1.2}$$

Fix a non-empty open set $U \subseteq X$. As x is a transitive point, there exist δ and $k \in \mathbb{Z}_+$ such that $T^k B_\delta(x) \subseteq U$. So, there exists $y \in B_\delta(x)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} d(T^i x, T^i y) > \varepsilon.$$

Let $z = T^k x$ and $z' = T^k y$. Then $z, z' \in U$ and

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} d(T^i z, T^i z') &= \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} d(T^i(T^k x), T^i(T^k y)) \\ &= \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} d(T^i x, T^i y) > \delta.\end{aligned}$$

Therefore, (X, T) is mean sensitive.

Now, if x is a transitive point but does not satisfies (1.2), then x is a mean equicontinuity point. So, by the Lemma 1.12, is almost mean equicontinuous. $\square$

Mean equicontinuity is very related to measurable dynamics properties which we will not explore in this theses. While studying these characterizations, a natural stronger property appeared, diam-mean equicontinuity. We will also study this property on this theses.

Definition 1.15. Let (X, T) be a TDS.

- We say that $x \in X$ is a **diam-mean equicontinuity point** of (X, T) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \operatorname{diam}(T^i B_\delta(x))}{n} < \varepsilon.$$

  We denote the set of diam-mean equicontinuity points of (X, T) by EQ^{DM}.

- The TDS (X, T) is **diam-mean equicontinuous** if $EQ^{DM} = X$.

- The TDS (X, T) **almost diam-mean equicontinuous** is EQ^{DM} is a residual set.

- The TDS (X, T) is **diam-mean sensitive** if there exists $\varepsilon > 0$ such that for every open set U we have

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \operatorname{diam}(T^i U)}{n} > \varepsilon.$$

The point 0^∞ in Example 1.11 is a mean equicontinuity point but it is not a diam mean equicontinuity point. On the other hand, the Sturmian shift is diam-mean equicontinuous.

The proof of the next lemma is very similar to the proof of Proposition 1.12; it only differs by some technical changes. However, the proof will be added to make it even clearer.

Lemma 1.16. *[13, Lemma 5]*
Let (X, T) be a TDS and $\varepsilon > 0$. Define

$$EQ^{DM}_\varepsilon = \{x \in X : \exists\, \delta > 0,\ \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \operatorname{diam}(T^i B_\delta(x))}{n} < \varepsilon\}. \tag{1.3}$$

Then, EQ^{DM}_ε is open and $EQ^{DM} = \bigcap_{m>0} EQ^{DM}_{\frac{1}{m}}$. Furthermore,if EQ^{DM} is nonempty, then EQ^{DM} is dense if and only if it is a residual set.

Proof. Let $x \in EQ_\varepsilon^{DM}$; that is, there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n \operatorname{diam}(T^i B_\delta(x))}{n} < \varepsilon.$$

Also, for every $y \in B_\delta(x)$ there exists $\delta' > 0$ satisfying that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n \operatorname{diam}(T^i B_{\delta'}(y))}{n} < \varepsilon.$$

Therefore, EQ_ε^{DM} is open.

Next, we show that $EQ^{DM} = \bigcap_{m>0} EQ^{DM}_{\frac{1}{m}}$.

- $\subseteq$: Let $x \in EQ^{DM}$ and $\varepsilon > 0$. By hypothesis, there exists $\delta > 0$ such that for every $p \in B_\delta(x)$ we have that:

  $$\limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i x, T^i p)}{n+1} < \frac{\varepsilon}{2}.$$

  Now, let $y, z \in B_\delta(x)$. We have that:

  $$\begin{aligned} & \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i y, T^i z)}{n+1} \\ \leq\ & \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i y, T^i x) + \sum_{i=0}^n d(T^i x, T^i z)}{n+1} \\ \leq\ & \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i x, T^i y)}{n+1} + \limsup_{n\to\infty} \frac{\sum_{i=0}^n d(T^i x, T^i z)}{n+1} \\ <\ & \varepsilon. \end{aligned}$$

  Therefore $x \in \bigcap_{m>0} EQ^{DM}_{\frac{1}{m}}$.

- $\supseteq$: This part is straightforward to show.

By the Baire theorem, we have that EQ^{DM} is dense if and only if it is a residual set. □

As the Lemma 1.16 is analogous to the Proposition 1.12, the following lemma is analogous to Lemma 1.13.

Lemma 1.17. *[13, Lemma 46]*
Let (X, T) be a TDS. The sets EQ^{DM} and EQ_ε^{DM} are inversely invariant, i.e., $T^{-n}EQ^{DM} \subseteq EQ^{DM}$, $T^{-n}EQ_\varepsilon^{DM} \subseteq EQ_\varepsilon^{DM}$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. Let $x \in X$ with $Tx \in EQ_\varepsilon^{DM}$. Choose $\delta > 0$ satisfying (1.3) for Tx. By the continuity of T, there exists $\eta > 0$ such that $d(Tx, Ty) < \delta$ for any $y \in B_\eta(x)$. Thus

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \operatorname{diam}(T^i B_\eta(x)) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \operatorname{diam}(T^i B_\delta(Tx)) < \varepsilon.$$

This implies that $x \in EQ_\varepsilon^{DM}$.

Observe that for any $\varepsilon' > \varepsilon > 0$ we have that $EQ_\varepsilon^{DM} \subseteq EQ_{\varepsilon'}^{DM}$. Hence, $T^{-n}EQ_\varepsilon^{DM} \subseteq T^{-n}EQ_{\varepsilon'}^{DM}$ for all $n \in \mathbb{Z}_{\geq 0}$. From this observation, we have that $T^{-n}EQ^{DM} = \bigcap_{m\geq 0} T^{-n}EQ^{DM}_{2^{-m}} \subseteq \bigcap_{m\geq 0} EQ^{DM}_{2^{-m}} = EQ^{DM}$. □

Theorem 1.18. *[13, Proposition 57]*

If (X,T) is transitive TDS, then it is either almost diam-mean equicontinuous or diam-mean sensitive.

Proof. Let $x \in X$ be a transitive point such that there exists $\varepsilon > 0$ such that every $\delta > 0$:

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \operatorname{diam}(T^i B_\delta(x))}{n} > \varepsilon. \tag{1.4}$$

Fix a non-empty open set $U \subseteq X$. As x is a transitive point, there exist $\delta > 0$ and $k \in \mathbb{Z}_+$ such that $T^k B_\delta(x) \subseteq U$. Hence,

$$T^i(T^k B_\delta(x)) \subseteq T^i U.$$

This implies that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \operatorname{diam}(T^i U)}{n} > \varepsilon$$

Therefore (X,T) is diam-mean sensitive.

Now, if x is a transitive point but does not satisfy (1.4), then x is a diam-mean equicontinuity point. So, by Lemma 1.16, (X,T) is almost diam-mean equicontinuous. □

In summary, we have the following notions which are related with strict implications.

Equicontinuity	$\Rightarrow$	Diam-mean equicontinuity	$\Rightarrow$	Mean equicontinuity;
Mean sensitivity	$\Rightarrow$	Diam-mean sensitivity	$\Rightarrow$	Sensitivity.

Other notions that will be of interest in this work are the following. In the introduction we talked about "largeness" of the set of all $n \in \mathbb{N}$ where this "significant separation" happens for a given region of X. We could have that the complement of this set is finite (**cofinitely sensitive**). There are bounded gaps of consecutive entries of the complement occur (**syndetically sensitive**). The following definition formulates these notions of sensitivity formally.

Definition 1.19. Let (X,T) be a TDS. For $U \subset X$ and $\delta > 0$, let

$$N_T(U,\delta) := \{n \in \mathbb{N} : \operatorname{diam}(T^n U) > \delta\}.$$

- We say that (X,T) is **cofinitely sensitive**, if there exists $\delta > 0$ such that for every nonempty open set $U \subset X$, we have that $N_T(U,\delta)$ is cofinite; that is: $\mathbb{N} \setminus N_T(U,\delta)$ is finite.
- We say that (X,T) is **syndetically sensitive**, if there exists $\delta > 0$ with the property that for every nonempty open set $U \subset X$, we have that $N_T(U,\delta)$ is syndetic; that is: $\mathbb{N} \setminus N_T(U,\delta)$ does not contain arbitrarily large blocks of consecutive integers.

Chapter 2

Cellular Automata

In this chapter, we introduce cellular automata (CA). Typically, cellular automata are defined on a full shift. We give a more general definition. This class of systems are also known as **shift-endomorphisms** or **sliding block-codes**. We use Theorem 2.2 to give an equivalent definition of CA. We will see that a CA is almost equicontinuous if and only if is not sensitive (see Proposition 2.8). As you can see transitivity of the CA is not required, but we do need the CA to commute with a transitive map. In Proposition 2.12 we will show how equicontinuity and periodicity are related locally.

Definition 2.1. We say that (X, T) is a **cellular automaton (CA)** if X is a subshift and $T : X \to X$ is a continuous map such that commutes with σ; i.e., $\sigma \circ T = T \circ \sigma$.

Cellular automata can be described using local rules. Note that Tx_i represents the ith coordinate of the point Tx.

Theorem 2.2 (Curtis-Hedlund-Lyndon). *Let X be a subshift and $T : X \to X$ a function. Then, (X, T) is a cellular automaton if and only if there exist integers $m \leq a$ and a (local) function $f : A^{a-m+1} \to A$ such that for any $x \in X$ and any $i \in \mathbb{Z}$*

$$Tx_i = f(x_{[i+m,i+a]}).$$

Proof. Let $T : X \to X$ be a CA. Let $r := \max\{-m, a\}$. Since $-r \leq m \leq a \leq r$, for any $n \geq 0$ we have

$$\begin{aligned} d(x, y) < 2^{-n-r} \quad &\Rightarrow \quad x_{[-n-r,n+r]} = y_{[-n-r,n+r]} \quad \Rightarrow \quad x_{[-n+m,n+a]} = y_{[-n+m,n+a]} \\ &\Rightarrow \quad Tx_{[-n,n]} = Ty_{[-n,n]} \quad \Rightarrow \quad d(Tx, Ty) < 2^{-n}. \end{aligned}$$

So, T is continuous. For any $i \in \mathbb{Z}$,

$$T\sigma(x)_i = f(\sigma x_{[i+m,i+a]}) = f(x_{[i+m+1,i+a+1]}) = Tx_{i+1} = \sigma(Tx)_i.$$

So, T commutes with the shift.

Conversely, we assume that T is a continuous map such that commutes with the shift. Since T is uniformly continuous, for $\varepsilon = 1$ there exists $r \geq 0$ such that

$$\begin{aligned} d(x, y) < 2^{-r} \quad &\Rightarrow \quad d(Tx, Ty) < 1 \\ x_{[-r,r]} = y_{[-r,r]} \quad &\Rightarrow \quad Tx_0 = Ty_0. \end{aligned}$$

There exists $f : A^{2r+1} \to A$ such that for any $x \in X$, $Tx_0 = f(x_{[-r,r]})$. Since T commutes with the shift,

$$Tx_i = \sigma(Tx)_0 = T(\sigma^i(x)_{[-r,r]}) = f(x_{[i-r,i+r]}).$$

Thus we have a local rule with $m = -r$ and and $a = r$. $\square$

The numbers m, a and r introduced in Theorem 2.2 and its proof are called **memory, anticipation** and **radius**, respectively. Cellular automata defined in $\{0, 1\}$ and radius $r = 1$ are called **elementary**. Their local rules are listed by numbers between 0 and 255 [21].

Example 2.3. The identity (Rule 204)
Let $(\{0, 1\}^{\mathbb{Z}}, I)$, where $Ix = x$.

Example 2.4. The zero map (Rule 0)
Let $(\{0, 1\}^{\mathbb{Z}}, O)$, where $Ox = 0^\infty$.

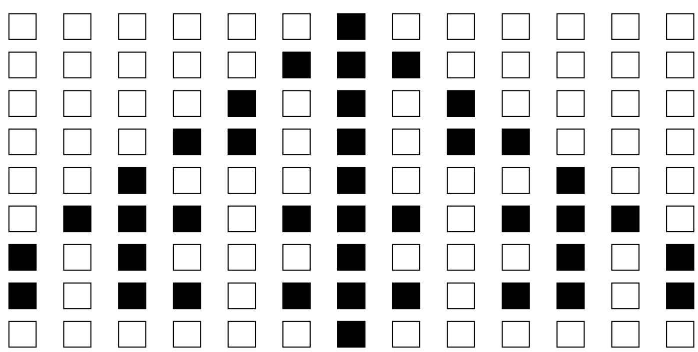
The reader can check that the previous CA are equicontinuous.

Example 2.5. The traffic CA (Rule 184)
Let $(\{0, 1\}^{\mathbb{Z}}, T)$ where

$$Tx_i = 1 \Leftrightarrow x_{[i-1,i]} = 10 \text{ or } x_{[i,i+1]} = 11.$$

This CA is sensitive.

Example 2.6. Rule 150
Let $A = \{0, 1\}$ and $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by $Tx_i = \text{mod}_2(x_{i-1} + x_i + x_{i+1})$. Let us represent $0 := \square$ and $1 := \blacksquare$. Let $x \in A^{\mathbb{Z}}$ such that $x_0 = \blacksquare$ and $x_i = \square$ for all $i \neq 0$. We have that



As we will see this example is permutative (and thus mean sensitive).

Example 2.7. The majority CA (Rule 232)
Let $(\{0, 1\}^{\mathbb{Z}}, M)$, where

$$Mx_i = \lfloor \frac{x_{i-1} + x_i + x_{i+1}}{2} \rfloor,$$

is almost equicontinuous. Notice that 0^∞ and 1^∞ are equicontinuity points, but $(01)^\infty$ is not.

In general, a CA satisfies the Akin-Auslander-Berg dichotomy without assuming transitivity, as it is proved in [21] for CA on the full shift. Using the same techniques we prove the following result for CA on transitive subshifts.

Proposition 2.8. *Let (X, σ) be a transitive subshift and (X, T) a CA. Then, (X, T) is almost equicontinuous if and only if is not sensitive.*

Proof. $\Rightarrow$: Assume that (X,T) is an almost equicontinuous CA. This means that for every open subset $U \subseteq X$, there exists $x \in U$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x,y) < \delta$, then we have that $d(T^n x, T^n y) < \varepsilon$ for all $n \geq 0$.

Let $\varepsilon > 0$. Observe that there exists $\delta > 0$ such that for all $y,z \in B_\delta(x)$ and all $n \geq 0$, we have that

$$\begin{array}{rcl} d(T^n y, T^n z) & \leq & d(T^n y, T^n x) + d(T^n x, T^n z) \\ & < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{array}$$

Therefore, (X,T) is not sensitive.

$\Leftarrow$: Assume that (X,T) is not sensitive; i.e., for all $\varepsilon > 0$ there exists an open set $U \subseteq X$ such that for all $x,y \in U$ and for all $n \geq 0$, we have that $d(T^n x, T^n y) < \varepsilon$. Now, since T is uniformly continuous, for $\varepsilon = 1$, there is $r \geq 0$ such that if $d(x,y) = 2^{-r}$, then $d(Tx,Ty) < 1$. This implies that for all $x,y \in X$ such that $x_{[-r,r]} = y_{[-r,r]}$, we have that $Tx_0 = Ty_0$. Hence, for all $m \geq 0$, there exist $d \geq r$ and $w \in A^{2d+1}$ such that for all $x,y \in X$ with $x_{[-d,d]} = w = y_{[-d,d]}$ and for all $n \geq 0$:

$$T^n x_{[-m,m]} = T^n y_{[-m,m]}.$$

Then, there is $p \in \{0,\ldots,|w|-r\}$ such that for all $x,y \in X$ satisfying $x_{[0,|w|-1]} = w = y_{[0,|w|-1]}$, we have

$$T^n x_{[p,p+r-1]} = T^n y_{[p,p+r-1]}$$

for all $n \in \mathbb{N}$.

For every $k \geq 0$ we define the set

$$\Omega_k = \{x \in X : \exists i \leq -k, x_{[i,i+|w|-1]} = w \wedge \exists j \geq k, x_{[j,j+|w|-1]} = w\}. \tag{2.1}$$

It is clear that the sets Ω_k are open. Furthermore, the transitivity of (X,T) implies Ω_K are non-empty and dense, for every $k \geq 0$. Therefore, $\bigcap_{k\geq 0} \Omega_k$ is a residual set. We show now that for every $m \geq 0$ there exists $k_m \geq 0$ such that

$$\Omega_{k_m} \subseteq EQ_{2^{-m}}.$$

Notice that for all $x,y \in \Omega_k$ we have:

$$T^n x_{[i+p,i+p+r-1]} = T^n x_{[j+p,j+p+r-1]} \text{ and } T^n y_{[i+p,i+p+r-1]} = T^n y_{[j+p,j+p+r-1]}.$$

If $x_{[i,j+|w|]} = y_{[i,j+|w|]}$, then for all $n \geq 0$ we obtain

$$T^n x_{[i+p,j+p+r-1]} = T^n y_{[i+p,j+p+r-1]}.$$

Therefore, for every $m \geq 0$, there exists $k_m \geq 0$ sufficiently large such that $\Omega_{k_m} \subseteq EQ_{2^{-m}}$. Hence, $\bigcap_{k_m\geq 0} \Omega_{k_m} \subseteq \bigcap_{m\geq 0} EQ_{2^{-m}}$. Thus, $\bigcap_{m\geq 0} EQ_{2^{-m}}$ is a residual set. Since $EQ = \bigcap_{m\geq 0} EQ_{2^{-m}}$, we conclude that (X,T) is an almost equicontinuous CA.

□

Example 2.9. Let us define $X = \{0,1\}^{\mathbb{Z}} \cup \{2,3\}^{\mathbb{Z}}$. Let $T : X \to X$ as

$$Tx = \begin{cases} \sigma x & if \quad x \in \{0,1\}^{\mathbb{Z}} \\ x & if \quad x \in \{2,3\}^{\mathbb{Z}}, \end{cases}$$

where σ is the shift function. Every point $x \in \{2,3\}^{\mathbb{Z}}$ is an equicontinuity point. For all non-empty open set $U \subset \{0,1\}^{\mathbb{Z}}$ there exist $x,y \in U$ and $n > 0$ such that $d(T^n x, T^n y) = 1$. Hence, the CA (X,T) has equicontinuity points but is not almost equicontinuous.

A TDS is **minimal** if every orbit is dense. The Auslander-Yorke dichotomy states that every minimal TDS is either equicontinuous or sensitive [2]. Now, consider the proof of Proposition 2.8. Note that if (X, σ) is minimal then $\Omega_k = X$ (see (2.1)). Using this observation we obtain the following result:

Proposition 2.10. *Let (X, σ) be a minimal subshift and (X, T) a CA. Then, (X, T) is equicontinuous if and only if is not sensitive.*

As we will see equicontinuity is a very strong property on CA because it implies eventual periodicity on the columns.

Definition 2.11. Let $(A^{\mathbb{Z}}, T)$ be a CA. A point $x \in A^{\mathbb{Z}}$ is m**-locally eventually point of** T, where $m > 0$, if $T^i x_{[-m,m]}$ is eventually periodic, with respect to i. We call x **locally eventually periodic point** if it is m-locally eventually point of T for every $m > 0$.

Proposition 2.12. *Let $(A^{\mathbb{Z}}, T)$ be a CA. Any equicontinuity point in $A^{\mathbb{Z}}$ is locally eventually periodic.*

Proof. If the set $EQ \neq \emptyset$, then $(A^{\mathbb{Z}}, T)$ is almost equicontinuous. Thus, for every $\varepsilon > 0$ exists a open set $U \subset A^{\mathbb{Z}}$ such that for all $x, y \in U$ and all $n \geq 0$, we have that $d(T^n x, T^n y) < \varepsilon$. Since $(A^{\mathbb{Z}}, \sigma)$ is transitive, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(T^n x, T^n y) < \varepsilon$.

Let $x \in EQ$. For all $m \neq 0$, such that $\delta \geq 2^{-m}$, there exist $j > j' \geq 0$ such that $T^j x_{[-m,m]} = T^{j'} x_{[-m,m]}$. Hence, $d(T^{j+i}x, T^{j'+i}x) < \delta$ for all $i \geq 0$. So, there exists $p \geq 0$ such that $j + p = j'$. Then, this implies that $T^j x_{[-m',m']} = T^{j+p} x_{[-m',m']} = T^{j'+p} x_{[-m',m']}$ for all $m' \geq 0$ such that $\varepsilon \geq 2^{-m'}$. Therefore, x is locally eventually periodic.

□

One can easily construct mean sensitive CA using the permutative CA.

Definition 2.13. Let $(A^{\mathbb{Z}}, T)$ be a CA with local rule $f : A^{a-m+1} \to A$.

- T is **left-permutative** if for all $u \in A^{a-m}$ and for all $b \in A$, there exists a unique $c \in A$ such that $f(cu) = b$.
- T is **right-permutative** if for all $u \in A^{a-m}$ and for all $b \in A$, there exists a unique $c \in A$ such that $f(uc) = b$.
- T is **permutative** if it is either left-permutative or right-permutative.

The next result follows immediately from Definition 2.13.

Lemma 2.14. *Let $(A^{\mathbb{Z}}, T)$ be a CA. For each $u \in A^{a-m}$ we define the function $f_u : A \to A$ as $f_u(c) = f(cu)$ (or $f_u(c) = f(uc)$). If $(A^{\mathbb{Z}}, T)$ is a left-permutative (right-permutative) CA, then for all $u \in A^{a-m}$, the function $f_u : A \to A$ bijective.*

Proposition 2.15. *If $(A^{\mathbb{Z}}, T)$ is permutative, then for every $x \in A$ and for every $M > 0$ there exists $y \in [x_{[-M,M]}]$ satisfying:*

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} = 1.$$

Proof. Let $x \in A^{\mathbb{Z}}$ and let us assume that $(A^{\mathbb{Z}}, T)$ is right-permutative. We are going to construct an element $y \in A^{\mathbb{Z}}$ such that $d(T^i x, T^i y) = 1$ for every $i \geq 1$. Without losing generality, fix $y_{(\infty,a)} = x_{(\infty,a)}$. Then, there exists $c_1 \in A$ such that $c_1 \neq x_{a-1}$ and $f(x_{[m,a-1]}c_1) \neq f(x_{[m,a]})$. Fix $y_{[a,2a-1]} = c_1 x_{[a+1,2a-1]}$. Since $Tx_0 \neq Ty_0$, Lemma 2.14 implies that there exists $u \in A$ such that $f(Tx_{[m,a-1]}u) \neq f(Tx_{[m,a]})$. Then, there exists $c_2 \in A$ such that $f(y_{[1+m,2a-1]}c_2) = u$. Hence, $T^2 x_0 \neq T^2 y_0$.

Now, let us assume that for some $l > 0$, there exists $(c_i)_{i=1}^{l} \subseteq A$ such that $y_{la} = c_l$, $y_{[la+1,(l+1)a-1]} = x_{[la+1,(l+1)a-1]}$ and $T^i x_0 \neq T^i y_0$ for all $1 \leq i \leq l$. By Lemma 2.14, we have that there is $u \in A$ such that $f(T^l x_{[m,a-1]}u) \neq f(T^l x_{[m,a]})$. Since for such l is true, then there exists $c_{l+1} \in A$ such that $y_{(l+1)a} = c_{l+1}$, $y_{[(l+1)a+1,(l+2)a-1]} = x_{[(l+1)a+1,(l+2)a-1]}$ and $T^i x_0 \neq T^i y_0$ for all $1 \leq i \leq l+1$.

Thus, we have that there exists $(c_i)_{i=1}^{\infty} \subseteq A$ such that $y_{ia} = c_i$, $y_{[ia+1,(i+1)a-1]} = x_{[ia+1,(i+1)a-1]}$ and $T^i x_0 \neq T^i y_0$. Therefore,

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} = 1.$$

The proof for left-permutative CA is analogous.

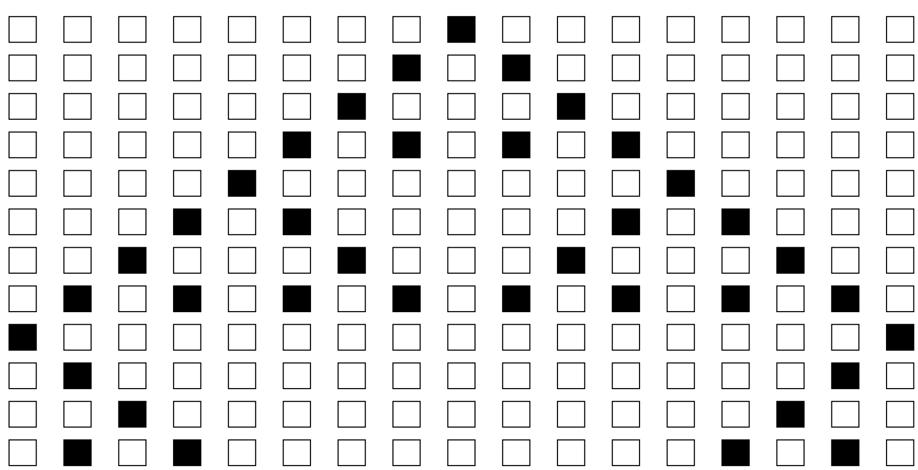
□

The proof of the following statement follows directly from Proposition 2.15.

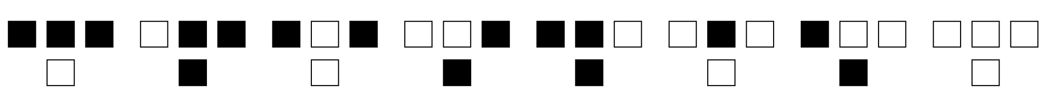
Proposition 2.16. *Any permutative CA $(A^{\mathbb{Z}}, T)$ is mean sensitive.*

Example 2.17. The sum rule 90

Let $(\{0,1\}^{\mathbb{Z}}, S)$, where $Sx_i = \mathrm{mod}_2(x_{i-1} + x_{i+1})$. We will show a section of the orbit of $x := {}^{\infty}\square\,\blacksquare\,\square^{\infty}$.



For this CA, the local rule is as follows:



From the local rule we can see easily that $(\{0,1\}^{\mathbb{Z}}, S)$ is permutative. Hence, by Proposition 2.16 we have that $(\{0,1\}^{\mathbb{Z}}, S)$ is mean sensitive. Moreover, by Proposition 2.15, $(\{0,1\}^{\mathbb{Z}}, S)$ is cofinitely sensitive.

Clearly, every almost equicontinuous TDS is almost mean equicontinuous. There exist many almost mean equicontinuous TDSs that are not almost equicontinuous (see Example 1.11)[22, 15]. However, none of the examples contained in [22, 15] is a CA. We will later construct an almost mean equicontinuous CA that is not almost equicontinuous.

Definition 2.18. Let $S \subseteq \mathbb{Z}_{\geq 0}$. We define the **upper density** of S by

$$\overline{D}(S) = \limsup_{n\to\infty} \frac{\sharp(S \cap \{0, \dots n-1\})}{n}.$$

Definition 2.19. Let $x \in X$, $J \subset \mathbb{Z}$ be finite set and $n \in \mathbb{N}$. For every $y \in B_n(x)$ we define

$$S_J = \{i \in \mathbb{N} : T^i y_J \neq T^i z_J\}.$$

For every pair of integers $n, k \in \mathbb{N}$ and every $x \in X$ we define the set

$$S_k^n(x, m) = S_{[-k,k]} \cap [0, n].$$

In view of Kurka's dichotomy; it is natural to ask if there is a mean (diam-mean) version of Theorem 2.8. Later on, we show that this question has a negative answer. First, we will give a more concrete characterization of mean (diam-mean) equicontinuity on CA. The following propositions use standard tools to connect density and averages.

Proposition 2.20. *Let (X, T) be a CA and $x \in X$. Then, x is a mean equicontinuity point if and only if for every $m \geq 0$ there exists $m' \geq 0$ such that for every $y \in B_{m'}(x)$ satisfies that*

$$\overline{D}(S_{\{-j,j\}}) \leq \frac{1}{2^{m+2}},$$

for all $0 \leq j \leq m+1$.

Proof. $\Rightarrow$: Let us assume that there exists $m \geq 0$ such that for all $m' \geq 0$ there exists $y \in B_{m'}(x)$ such that $\overline{D}(S_{\{-l,l\}}) > \frac{1}{2^{m+2}}$ for some $0 \leq l \leq m+1$. This implies that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} &\geq \limsup_{n\to\infty} \frac{\sum_{i\in S_{\{-l,l\}}\cap[0,n]} d(T^i x, T^i y)}{n+1} \\
&\geq \frac{1}{2^l} \limsup_{n\to\infty} \frac{\sharp(S_{\{-l,l\}} \cap [0, n])}{n+1} \\
&\geq \frac{1}{2^{m+1}} \overline{D}(S_{\{-l,l\}}) > \frac{1}{2^{2m+3}}.
\end{aligned}$$

Therefore, x is not a mean equicontinuity point.

$\Leftarrow$: Observe that for every k we have that

$$S_k^n \subseteq S_{k+1}^n$$

and

$$S_{k+1}^n \setminus S_k^n \subseteq S_{\{-(k+1),k+1\}} \cap [0, n]$$

Now, let us assume that, for every $m \geq 0$ there exists $m' \geq 0$ such that every $y \in B_{m'}(x)$, satisfies $\overline{D}(S_{\{-j,j\}}) \leq \frac{1}{2^{m+2}}$, for every $0 \leq j \leq m+1$. Then,

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} \\
= & \limsup_{n\to\infty} \frac{\sharp(S_0^n) + \sum_{i=1}^{\infty} \frac{1}{2^i}\sharp(S_i^n \setminus S_{i-1}^n)}{n+1} \\
\leq & \sum_{i=0}^{m+1} \frac{1}{2^i}\overline{D}(S_{\{-i,i\}}) + \sum_{i=m+2}^{\infty} \frac{1}{2^i}\overline{D}(S_{\{-i,i\}}) \\
\leq & \frac{1}{2^{m+2}} \sum_{i=0}^{m+1} \frac{1}{2^i} + \sum_{i=m+2}^{\infty} \frac{1}{2^i} \\
\leq & \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} = \frac{1}{2^m}.
\end{aligned}$$

This implies x is a mean equicontinuity point. □

Now we will define sensitivity sets on a set of columns.

Definition 2.21. Let $J \subset \mathbb{Z}$ be finite set and $n \in \mathbb{N}$. We define

$$S_J^{DM}(x,n) = \{i \in \mathbb{N} : \exists\ y, z \in B_n(x), T^i y_J \neq T^i z_J\}.$$

For every pair of integers $n, k \in \mathbb{N}$ and every $x \in X$ we define the set

$$S_k^n(x,m) = S_{[-k,k]}^{DM}(x,m) \cap [0,n].$$

Proposition 2.22. *Let (X,T) be a CA and $x \in X$. Then x is a diam-mean equicontinuity point if and only if for every $m \geq 0$ there exists $m' \geq 0$ such that*

$$\overline{D}(S_{\{-j,j\}}^{DM}(x,m')) \leq \frac{1}{2^{m+2}}$$

for all $0 \leq j \leq m+1$.

Proof. $\Rightarrow$: Suppose there exists $m \geq 0$ such that for all $m' \geq 0$ there exists $l \in [0, m+1]$ such that

$$\overline{D}(S_{\{-l,l\}}^{DM}(x,m')) > \frac{1}{2^{m+1}}.$$

This implies that

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} \operatorname{diam}(T^i B_{m'}(x))}{n+1} \geq \limsup_{n\to\infty} \frac{\sum_{i\in S_{\{-l,l\}}^{DM}(x,m')\cap[0,n]} \operatorname{diam}(T^i B_{m'}(x))}{n+1}.$$

$$\begin{aligned}
&\geq \frac{1}{2^l} \limsup_{n\to\infty} \frac{1}{n+1}\sharp(S_{\{-l,l\}}^{DM}(x,m') \cap [0,n]) \\
&\geq \frac{1}{2^{m+1}}\overline{D}(S_{\{-l,l\}}^{DM}(x,m')) > \frac{1}{2^{2m+3}}.
\end{aligned}$$

Therefore, x is not a diam-mean equicontinuity point.

$\Leftarrow$: Note that for every k we have that

$$S_k^n(x,m) \subseteq S_{k+1}^n(x,m),$$

and

$$S_{k+1}^n(x,m) \setminus S_k^n(x,m) \subseteq S_{\{-(k+1),k+1\}}^{DM}(x,m) \cap [0,n].$$

Now, let us assume that for every $m \geq 0$ there exists $m' \geq 0$ such that

$$\overline{D}(S_{\{-j,j\}}^{DM}(x,m')) \leq \frac{1}{2^{m+2}}$$

for every $0 \leq j \leq m+1$. For sufficiently large m we conclude that

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \operatorname{diam}(T^i B_{m'}(x)) \\
\leq \; & \limsup_{n\to\infty} \frac{1}{n+1} \left[\sharp(S_0^n(x,m')) + \sum_{i=1}^{\infty} \frac{1}{2^i} \sharp(S_i^n(x,m') \setminus S_{i-1}^n(x,m')) \right] \\
\leq \; & \sum_{i=0}^{m+1} \frac{1}{2^i} \overline{D}(S_{\{-i,i\}}^{DM}(x,m')) + \sum_{i=m+2}^{\infty} \frac{1}{2^i} \overline{D}(S_{\{-i,i\}}^{DM}(x,m')) \\
\leq \; & \frac{1}{2^{m+2}} \sum_{i=0}^{m+1} \frac{1}{2^i} + \sum_{i=m+2}^{\infty} \frac{1}{2^i} \\
\leq \; & \frac{1}{2^m}.
\end{aligned}$$

This implies x is a diam-mean equicontinuity point. $\square$

Chapter 3

New examples

During this chapter we seek to answer the following questions:

- Is the set of almost equicontinuous CA properly contained in the set of almost diam-mean equicontinuous CA?
- Is the set of almost diam-mean equicontinuous CA properly contained in the set of almost mean equicontinuous CA?
- Is there a dichotomy for (diam-)mean sensitivity and (diam-)mean equicontinuity for CA (similar to Kurka's dichotomy)?
- Is there a CA that is almost mean equicontinuous and diam-mean sensitive?

Our objective is to construct four examples to answer the questions above. As an additional result, we have that the example given in Section 3.1 shows the existence of CA that are almost mean equicontinuous but cofinitely sensitive.

3.1 The Pacman CA

All the results in this section and their proofs are contained in [6]. We respect the order stated in the cited article within this section.

In this section, we will construct a CA that is almost mean equicontinuous but not almost equicontinuous. Firstly, we will give the formal definition of the CA, then we will give the heuristics of the map so the reader gets intuition and, finally, we will approach the result using a series of technical lemmas. We remind the reader that Tx_i represents the ith coordinate of the point Tx.

Let $A = \{\square, \boxed{d}, \boxed{g}, \boxed{k}, \boxed{p}, \boxed{dg}\}$. We define the function $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ locally as follows

$$Tx_i = \begin{cases} \square & if \quad (x_{i-1} \in \{\square, \boxed{g}, \boxed{k}\} \wedge [(x_i \in \{\square, \boxed{g}, \boxed{k}\} \wedge x_{i+1} \in \{\square, \boxed{d}, \boxed{p}\}) \\ & \qquad \vee (x_i = \boxed{p} \wedge x_{i+1} \notin \{\boxed{d}, \boxed{dg}\})]) \\ & \qquad \vee (x_{i-1} \in \{\boxed{d}, \boxed{dg}\} \wedge [(x_i \in \{\square, \boxed{k}\} \wedge x_{i+1} \in \{\square, \boxed{d}, \boxed{p}\}) \\ & \qquad \vee (x_i = \boxed{p} \wedge x_{i+1} \notin \{\boxed{d}, \boxed{dg}\})]), \\ \boxed{d} & if \quad x_i \in \{\boxed{d}, \boxed{dg}\} \wedge x_{i+1} \notin \{\boxed{k}, \boxed{dg}\}, \\ \boxed{g} & if \quad (x_{i-1} \in \{\square, \boxed{g}, \boxed{k}\} \wedge x_i \in \{\square, \boxed{g}, \boxed{k}\} \wedge x_{i+1} \in \{\boxed{g}, \boxed{dg}\}) \\ & \qquad \vee (x_{i-1} \in \{\boxed{d}, \boxed{dg}\} \wedge x_i \in \{\square, \boxed{k}\} \wedge x_{i+1} = \boxed{dg}), \\ \boxed{k} & if \quad (x_{i+1} = \boxed{k} \wedge [(x_{i-1} \in \{\square, \boxed{g}, \boxed{k}\} \wedge x_i \in \{\square, \boxed{g}, \boxed{k}\}) \\ & \qquad \vee (x_{i-1} \in \{\boxed{d}, \boxed{dg}\} \wedge x_i \in \{\square, \boxed{k}\})]) \\ & \qquad \vee (x_{i-1} = \boxed{p} \wedge x_i \notin \{\boxed{d}, \boxed{dg}\} \wedge x_{i+1} \in \{\boxed{d}, \boxed{dg}\}) \\ & \qquad \vee (x_i = \boxed{p} \wedge x_{i+1} \in \{\boxed{d}, \boxed{dg}\}), \\ \boxed{p} & if \quad (x_{i-1} = \boxed{d} \wedge [(x_i \in \{\square, \boxed{k}\} \wedge x_{i+1} = \boxed{g}) \vee x_i = \boxed{g}]) \\ & \qquad \vee (x_{i-1} = \boxed{p} \wedge x_i, x_{i+1} \notin \{\boxed{d}, \boxed{dg}\}), \text{ and} \\ \boxed{dg} & if \quad x_i \in \{\boxed{d}, \boxed{dg}\} \wedge x_{i+1} \in \{\boxed{k}, \boxed{dg}\}. \end{cases}$$

We call this CA the **Pacman cellular automaton**. Notice that this CA has memory and anticipation 1. We will call the members of the alphabet A as follows:

- $\square$ empty space,
- $\boxed{d}$ empty door,
- $\boxed{p}$ pacman,
- $\boxed{g}$ ghost.
- $\boxed{k}$ keymaster ghost, and
- $\boxed{dg}$ door with ghost.

We will now explain the heuristics of this map so the reader gets intuition on the dynamics. The reader does not need to know the rules of the game **Pacman**. It is only needed to understand that pacmans eat blue ghosts.

- A door always stays fixed in the same place (a ghost might cross it); that is, $x_i \in \{\boxed{d}, \boxed{dg}\}$ if and only if $Tx_i \in \{\boxed{d}, \boxed{dg}\}$.
- Pacmans $\boxed{p}$ move to right (one position per unit of time) if there is no door; that is, if $x_i = \boxed{p}$ and $x_{i+1}, x_{i+2} \notin \{\boxed{d}, \boxed{dg}\}$ then $Tx_{i+1} = \boxed{p}$.
- If a pacman encounters a door (on the right) it is transformed into keymaster ghost $\boxed{k}$; that is, if $x_i = \boxed{p}$ and $x_j \in \{\boxed{d}, \boxed{dg}\}$ with $j \in \{i+1, i+2\}$ then $Tx_{j-1} = \boxed{k}$.
- Ghosts ($\boxed{g}$,$\boxed{k}$) always move to the left (one position per unit of time) if there is no pacman or a door on the left; that is, if $x_i = \boxed{g}(x_i = \boxed{k})$, $x_{i-1} \in \{\square, \boxed{g}, \boxed{k}\}$ and $x_{i-2} \in \{\square, \boxed{g}, \boxed{k}\}(x_{i-2} \neq \boxed{p})$, then $Tx_{i-1} = \boxed{g}(Tx_{i-1} = \boxed{k})$.
- If a ghost or keymaster ghost encounters a pacman (on the left) it will disappear (get eaten); that is,
  - if $x_i \in \{\boxed{g}, \boxed{k}\}$ and $x_{i-1} = \boxed{p}$, then $Tx_{i-1} \notin \{\boxed{g}, \boxed{k}\}$; and

– if $x_i \in \{\ ,\ \}$, $x_{i-2} =$ and $x_{i-1} \notin \{\ ,\ \}$, then $Tx_{i-1} =$.

- If a ghost encounters a door it transforms into a pacman; that is,

  – if $x_i =$ and $x_{i-1} \in \{\ ,\ \}$, then $Tx_i =$; and

  – if $x_i =$, $x_{i-2} = \{\ ,\ \}$ and $x_{i-1} \notin \{\ ,\ ,\ \}$, then $Tx_{i-1} =$.

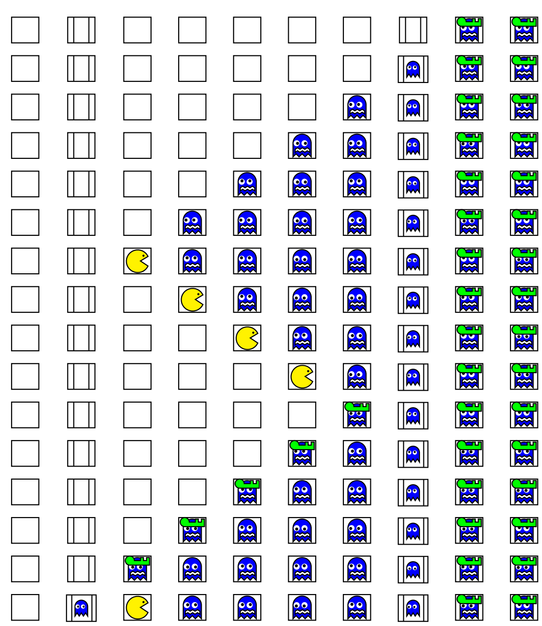
- If a keymaster ghost encounters a door he will enter the door, lose its key, and (in the following step) proceed to the left; that is, if $x_i =$ and $x_{i-1} \in \{\ ,\ \}$, then $Tx_{i-1} =$ and

  – if $x_{i-3} =$, then $T^2 x_{i-2} =$ and

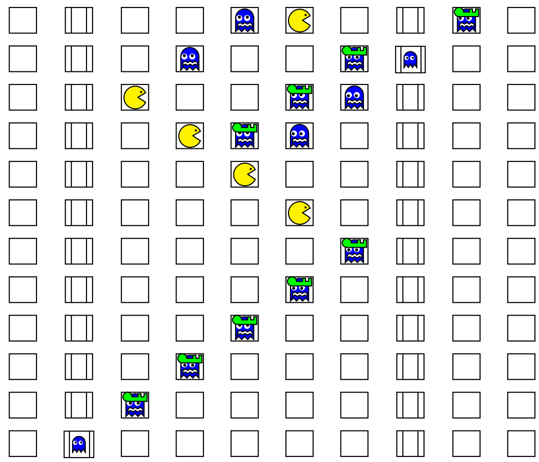
  – if $x_{i-3} =$, then $T^2 x_{i-2} =$.

When describing a point x in $A^{\mathbb{Z}}$ we will use a point (.) to indicate the **0th** coordinate of x. For example, if $x = {}^{\infty}\ .\ \ ^{\infty}$ then $x_0 =$ and $x_i =$ for every $i \neq 0$. We will now provide some examples on how the Pacman CA works. Notice that time flows downward on the diagrams.

Example 3.1. Let $m \geq 2$ and $w = \ \ ^m\ $. Let us show a section of the orbit of $x :={}^{\infty}\ .w\ ^{\infty}$. In this example, we can observe that the space between two doors is acting like a some sort of "filter", because many ghosts disappear.



Example 3.2. Let $w = \boxed{\|}\,\square\,\square$ 👻 ᗧ $\square\,\boxed{\|}$ 👾. We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



We now prove a series of technical lemmas. If there is one 👾 to the right of a pattern with empty spaces and empty doors, then 👾 will "cross" all the doors eventually. We state this fact formally in the next lemma.

Lemma 3.3. *Let $m \geq 1$, $w \in \{\square, \boxed{\|}\}^m$ such that $w_0 = \boxed{\|} = w_{m-1}$ and $w_i = \square$ for all $0 < i < m-1$. Set $x = {}^{\infty}\square.w$ 👾 $\square^{\infty}$. Then, there exists $N > 0$ such that $T^N x_{[0,m-1]} = w$.*

Proof. Assume that $m \geq 4$. From the definition of T we have the followings implications:

- $Tx_{m-1} =$ $\boxed{👻}$ $\wedge\ Tx_i \in \{\square, \boxed{\|}\}\ \forall\ i \neq m-1$,
- $T^{m-j}x_j =$ 👻 for $1 < j < m-1 \wedge T^{m-j}x_i \in \{\square, \boxed{\|}\}\ \forall\ i \neq j$,
- $T^{m-2+j}x_j =$ ᗧ for $1 \leq j < m-2 \wedge T^{m-2+j}x_i \in \{\square, \boxed{\|}\}\ \forall\ i \neq j$,
- $T^{3m-6-j}x_j =$ 👾 for $1 \leq j \leq m-2 \wedge T^{3m-6-j}x_i \in \{\square, \boxed{\|}\}\ \forall\ i \neq j$,
- $T^{3m-6}x_0 =$ $\boxed{👻}$ $\wedge\ T^{3m-6}x_i \in \{\square, \boxed{\|}\}\ \forall\ i \neq 0$.

Then, for $N = 3m-5$, we have that $T^N x_{[0,m-1]} = w$. The case when $1 \leq m \leq 3$ is easy to check. $\square$

One thing to notice about Examples 3.1 and 3.2 is that $\boxed{\|}$ stays fixed in the same place.

Remark 3.4. Note that if $x_i = \boxed{\|}$ and $x_{i+1} = \square$ then $Tx_i = \boxed{\|}$.

Using Remark 3.4 and Lemma 3.3 we obtain the following:

Lemma 3.5. *Let $m > 0$, $w \in A^m$ and $x = {}^{\infty}\square.w\boxed{\|}\,\square^{\infty}$. There exists $N > 0$ such that for all $n \geq N$,*

$$T^n x_i \in \{\square, \boxed{\|}\}\ \forall\ i \geq 0.$$

In the proof of Lemma 3.3 we describe the "trajectory" of 👾 from the start until it crosses the doors. In the following lemma, we describe a similar trajectory, but this time we are going to do it backwards in time.

Lemma 3.6. *Let $m \geq 2$, $v \in A^{\mathbb{N}}$ and $x := {}^{\infty}\square.\boxed{\|}\,\square^m\,\boxed{\|}v$. If $N \geq 3m$ and $T^N x_0 =$ $\boxed{👻}$, then*

- $T^{N-3m+1}x_{m+1} =$,
- $T^{N-2(m-1)-j}x_j =$ *for* $2 \leq j \leq m$*,*
- $T^{N-m-j}x_{m-j} =$ *for* $1 \leq j \leq m-1$*, and*
- $T^{N-j}x_j =$ *for* $1 \leq j \leq m$*.*

Proof. Assume the hypothesis of the lemma. By checking the rules of T we can see that if $T^N x_0 =$ and $x_1 \neq$ then necessarily $T^{N-1}x_1 =$. We can go back step by step to obtain the result. $\square$

Using Lemma 3.6, we will see, that if $T^N x_0 = \; = T^{N'}x_0$ then N and N' cannot be near.

Lemma 3.7. *Let* $m \geq 0$*,* $v \in A^{\mathbb{N}}$*, and* $x := {}^{\infty}$. ^{m} v*. If* $N' > N \geq 3m$ *are such that* $T^N x_0 = \; = T^{N'}x_0$*, then:*

- *if* $0 \leq m \leq 1$*, then* $N' - N > 2m$*; and*
- *if* $m \geq 2$*, then* $N' - N \geq 2m - 1$*.*

Proof. The case $0 \leq m \leq 1$ is straightforward to check.

Let $m \geq 2$, and $N' > N \geq 3m$ such that $T^N x_0 = \; = T^{N'}x_0$. Assume that $N' - N < 2m - 1$. From Lemma 3.6 we have that:

- $T^{N-j}x_j =$ for $1 \leq j \leq m$ and
- $T^{N'-m-j'}x_{m-j'} =$ for $1 \leq j' \leq m-1$.

First, suppose that $N' - N$ is even. Let $j = m - \frac{N'-N}{2}$ and $j' = \frac{N'-N}{2}$. By the assumption on N, N' and m it follows that $1 \leq j \leq m$, $1 \leq j' \leq m-1$, and

$$T^{N-j}x_j = T^{N'-m-j'}x_{m-j};$$

a clear contradiction.

Now suppose $N' - N$ is odd. Let $j = m - \lceil \frac{N'-N}{2} \rceil$ and $j' = \lceil \frac{N'-N}{2} \rceil$. By the assumption on N, N' and m it follows that $1 \leq j \leq m$, $1 \leq j' \leq m-1$,

$$T^{N-j}x_j = \; , \text{ and } T^{N'-m-j'}x_{m-j'} = \; .$$

Therefore,

$$T^{N-j}x_{[j,j+1]} = \; ;$$

which is also a contradiction because is not on the image of T.

$\square$

In Example 3.1, we see that considering an infinite right-tail of keymater ghosts, some get eaten and some cross the doors. It is natural to ask what will be the frequency of that cross a doors. The next lemma answers this question.

Lemma 3.8. *Let* $w =$ ^{m} *, with* $m \geq 0$ *and* $x = {}^{\infty}$ $.w$ $^{\infty}$*.*

If $0 \leq m \leq 1$*, then*

- $T^{3m-2}x_0 =$ ,
- $T^{3m-2+(2m+1)k}x_0 =$  *for* $k \geq 0$, *and*
- $T^i x_0 = \boxed{\square}$, *for all* $3m-2+(2m+1)k < i < 3m-2+(2m+1)(k+1)$ *and* $k \geq 0$.

If $m \geq 2$, *then*

- $T^{3m}x_0 =$ ,
- $T^{3m+(2m-1)k}x_0 =$  *for* $k \geq 0$, *and*
- $T^i x_0 = \boxed{\square}$, *for all* $3m+(2m-1)k < i < 3m+(2m-1)(k+1)$ *and* $k \geq 0$.

Proof. The proof for $0 \leq m \leq 1$ is similar to the proof when $m \geq 2$. So, we are only going to prove the result when $m \geq 2$. Using a similar argument of the proof of Lemma 3.3, we obtain that $T^{3m}x_0 =$  and $T^i x_0 = \boxed{\square}$ for all $0 < i < 3m$. Also, we have that $T^{2m-1}x_{m+2} =$ . Hence, $T^{5m-1}x_0 =$ .

We will proceed by induction on k. Let us assume that

$$T^{3m+(2m-1)l}x_0 =$$ .

Next, let $k = l+1$. By the induction hypothesis, we have that

$$T^{2m-1+(2m-1)l}x_{(m+2)} =$$ .

Hence, $T^{5m-1+(2m-1)l}x_0 =$ . Doing simple calculations we obtain

$$T^{3m+(2m-1)(l+1)}x_0 =$$ .

The proof of $T^i x_0 = \boxed{\square}$, for all $3m+(2m-1)k < i < 3m+(2m-1)(k+1)$ and $k \geq 0$, follows immediately from Lemma 3.7. $\square$

We will now prove that the set of equicontinuity points of the Pacman CA is empty.

Proposition 3.9. *Let* $m \geq 1$ *and* $w \in A^m$. *Then, there exist* $x, y \in A^{\mathbb{Z}}$ *such that*

$$x_{[0,m-1]} = w = y_{[0,m-1]}$$

and the set

$$S = \{i \in \mathbb{Z}_{\geq 0} : T^i x_0 \neq T^i y_0\}$$

is infinite.

Proof. Let $m \geq 1$, $w \in A^m$ and $x = {}^{\infty}\square.w\square^{\infty}$. Lemma 3.5 says that there exists $N > 0$ such that $T^n x_0 \in \{\square, \boxed{\square}\}$ for every $n \geq N$. Let $y = {}^{\infty}\square.w\boxed{\square}$ $^{\infty}$. By Lemma 3.8, the the set S is infinite. $\square$

Lemma 3.8 tells us the exact frequency of  crossing doors when a ponit x has a tail $^{\infty}$ to the right. If we do not have precise information on what is in the right, we may not have the exact frequency as in the Lemma 3.8. However, using Lemma 3.7, we will be able to obtain an upper bound.

Now we will explore a similar situation but with finitely many doors.

Lemma 3.10. *Let $\{d_i\}_{i=0}^{n}$ a finite set of non-negative integers, $v \in A^{\mathbb{N}}$,*

$$w = \boxed{|}\square^{d_0}\boxed{|}\square^{d_1}\cdots\boxed{|}\square^{d_n}\boxed{|},$$

$x = {}^{\infty}\square.wv$, and $0 \leq j < n + \sum_{i=0}^{n-1} d_i$. Assume that $T^N x_j = \boxed{\blacksquare} = T^{N'} x_j$ for some $N, N' \geq 0$. We have that

- *if $0 \leq d_n \leq 1$, then $|N - N'| > 2d_n$, and*

- *if $d_n \geq 2$, then $|N - N'| > 2(d_n - 1)$.*

Proof. The case where $n = 0$ is a direct application of Lemma 3.7. We will prove the other case by induction. Assume that for $n = p$ the result holds. Now, let $n = p + 1$. By the induction hypothesis, we have that if $x_j = \boxed{|}$, for all $1 \leq j \leq p + 1 + \sum_{i=0}^{p} d_i$, and $T^N x_j = \boxed{\blacksquare} = T^{N'} x_j$ for all $N, N' \geq 0$ then:

$$\begin{array}{llll} \text{if} & d_{p+1} \geq 2 & \text{then} & |N - N'| \geq 2(d_{p+1} - 1) \text{ or} \\ \text{if} & 0 \leq d_{p+1} \leq 1 & \text{then} & |N - N'| \geq 2d_{p+1}. \end{array}$$

Hence, the only thing left to show is that, for $x_0 = \boxed{|}$ and for all $N, N' \geq 0$ such that $T^N x_0 = \boxed{\blacksquare} = T^{N'} x_0$ we have that:

$$\begin{array}{llll} \text{if} & d_{p+1} \geq 2 & \text{then} & |N - N'| \geq 2(d_{p+1} - 1) \text{ or} \\ \text{if} & 0 \leq d_{p+1} \leq 1 & \text{then} & |N - N'| \geq 2d_{p+1}. \end{array}$$

For $0 \leq d_{p+1} \leq 1$ the result is easy to check. So, let us assume that $d_{p+1} \geq 2$. Also, let us assume that there exist $N, N' \geq 0$ such that $T^N x_0 = \boxed{\blacksquare} = T^{N'} x_0$. This means that there exist $N_0, N_0' \geq 0$ such that $T^{N_0} x_{d_0+1} = \boxed{\blacksquare} = T^{N_0'} x_{d_0+1}$ and $N_0' + r = N'$ and $N_0 + r = N$. Therefore,

$$2(d_{p+1} - 1) \quad \leq \quad |N' - N|.$$

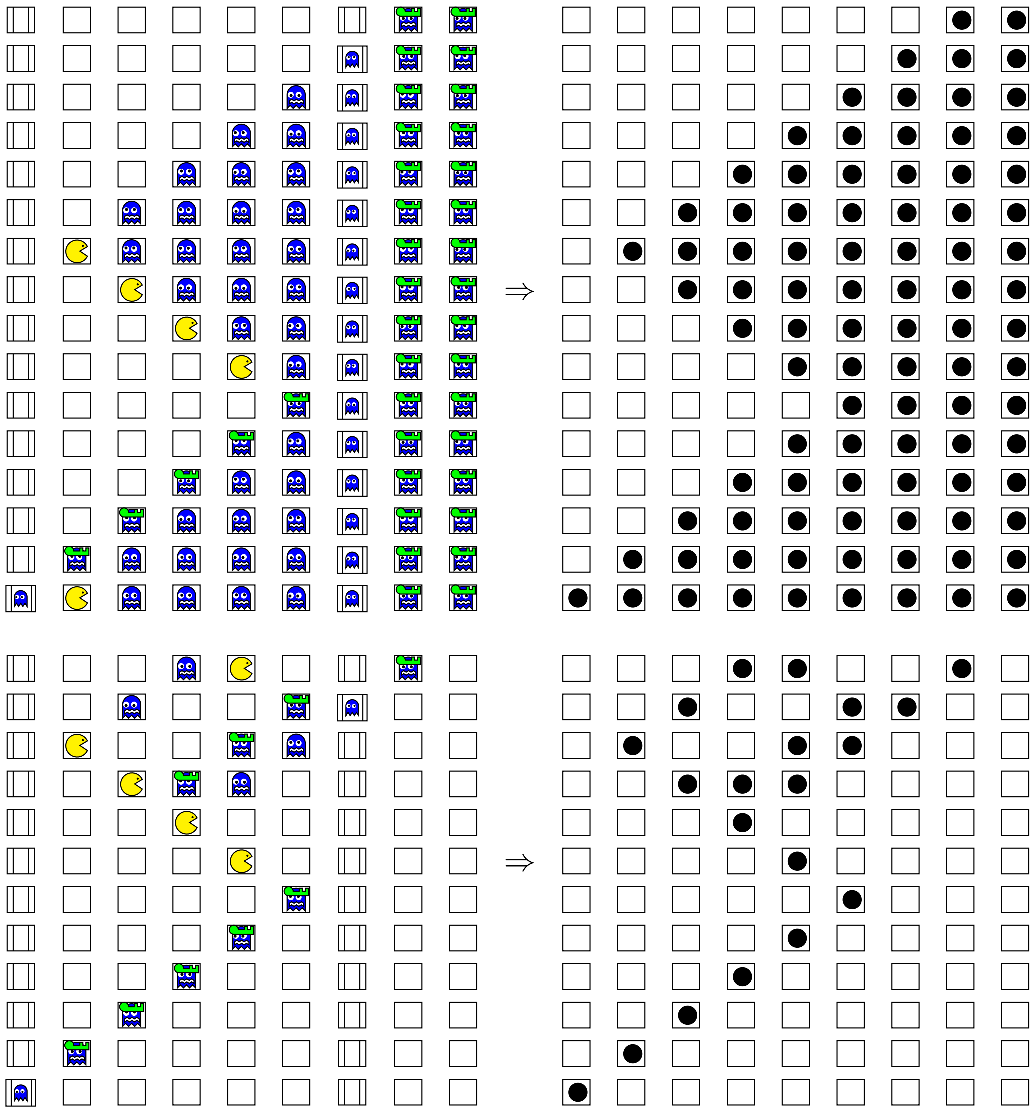
$\square$

For Lemma 3.11 it will be useful to consider the CA as a (vanishing) particle system, where ghosts and pacmans are particles.

We define the **particle function** $\gamma : A^{\mathbb{Z}} \to \{\square, \boxed{\bullet}\}^{\mathbb{Z}}$ as

$$\gamma(x)_i = \begin{cases} \square & if \quad x_i \in \{\square, \boxed{|}\}, \\ \boxed{\bullet} & if \quad x_i \in \{\blacksquare, \blacksquare, \blacksquare, \boxed{\blacksquare}\}, \end{cases}$$

where $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$. Observe that with this function the Examples 3.1 and 3.2 turn out as follows:



We can compare the density by columns for two different points in $A^{\mathbb{Z}}$.

Lemma 3.11. *Let $d > 0$, $w = \boxed{||}\,\square^d\boxed{||}$, $x :={}^\infty\square.w\square^\infty$, $v \in A^{\mathbb{N}}$, and $y ={}^\infty\square.wv$. If $1 \leq i \leq d$, then*

$$3\overline{D}(S_{\{d+1\}}) \geq \overline{D}(S_{\{i\}}).$$

Proof. Using the Pacman CA and the particle function, we can define a **trajectory function** of an specific particle (a ghost/pacman) p. We will not construct this function explicitly, but we will give its properties. Given a point $x \in X$ and a particle of that point; that is, $p \in \mathbb{Z}$ with $\gamma(x)_p = \boxed{\bullet}$, we can define trajectory of that particle (all the way to the infinity or until it disappears). This trajectory is a function $\tau_p : N \to \mathbb{Z}$ where $N \subset \mathbb{N}$ is the lifespan of the particle (i.e. $N = \mathbb{N}$ if it never disappears), $\tau_p(0) = p$ and $\tau_p(n)$ the position at time n. We have that $|\tau_p(n) - \tau_p(n+1)| \leq 1$ for $n+1 \in N$. Using the properties of T it is not hard to see that for every $z \in \mathbb{Z}$ we have that $|\tau_p^{-1}(z)| \leq 3$; that is, a particle can only be at most three times on a particular position. By Lemma 3.5, there exists $N > 0$ such that, if for some $l > 0$, $T^{N+l}y_i \notin \{\square, \boxed{||}\}$, then there exists a unique $k \geq |w|$, such that $\gamma(y)_k = \boxed{\bullet}$ and $\tau_k(N+l) = i$. Hence, for all $n \in S_{+i}$, there exist a unique $k_n \geq |w|$ and $m < n$ such that $\tau_{k_n}(n) = i$ and

$\tau_{k_n}(m) = d+1$. Since $y_{d+1} = \boxbar$, then $|\tau_{k_n}^{-1}(d+1)| = 1$. Define $P := \{z \in \mathbb{Z} : \gamma(y)_z = \boxed{\bullet}\}$. Therefore,

$$\limsup_{n\to\infty} \tfrac{3\sharp[(\bigcup_{z\in P} \tau_z^{-1}(d+1))\cap[0,n]]}{n+1} \quad \geq \quad \limsup_{n\to\infty} \tfrac{\sharp[(\bigcup_{z\in P} \tau_z^{-1}(i))\cap[0,n]]}{n+1}. \tag{3.1}$$

Since $\gamma(x)_z = \square$ for all $z \in \mathbb{Z}$, we have that

$$\bigcup_{z\in P} \tau_z^{-1}(i) = S_{\{i\}} \text{ and}$$

$$\bigcup_{z\in P} \tau_z^{-1}(d+1) = S_{\{d+1\}}.$$

Therefore, from (3.1), we conclude that

$$3\overline{D}(S_{\{d+1\}}) \geq \overline{D}(S_{\{i\}}).$$

$\square$

Notice that we know how ghosts behave between doors, how long it takes them to cross two walls, and how to compare column densities for two given elements in $A^{\mathbb{Z}}$. Using these information, we can construct an equicontinuity point for the Pacman CA.

Proposition 3.12. *Let* $x = \cdots \boxbar\,\square^{2^2}\boxbar\,\square^{2^1}\boxbar\,\square^{2^0}.\boxbar\,\square^{2^0}\boxbar\,\square^{2^1}\boxbar\,\square^{2^2}\cdots$. *Then, x is a mean equicontinuity point.*

Proof. We divide the proof in two parts:

Part 1: Let $m \geq 0$, $m' = m+3+\sum_{l=0}^{m+3} 2^l$, and $y \in X$ with

$$y_{[-k,k]} = \boxbar\,\square^{2^{m+3}}\cdots\boxbar\,\square^{2^2}\boxbar\,\square^{2^1}\boxbar\,\square^{2^0}.\boxbar\,\square^{2^0}\boxbar\,\square^{2^1}\boxbar\,\square^{2^2}\cdots\boxbar\,\square^{2^{m+3}}\boxbar,$$

for a certain k. By Lemma 3.10, we have that, if $x_j = \boxbar$, where $0 \leq j \leq m+3+\sum_{l=0}^{m+2} 2^l$, then for all $N, N' \geq 0$ such that $T^N y_j = \boxdot = T^{N'} y_j$, satisfies $|N - N'| \geq 2(2^{m+3}-1)$. Now,

$$\limsup_{n\to\infty} \frac{\sharp(S_{\{-j,j\}} \cap [0,n])}{n+1} \quad \leq \quad \limsup_{n\to n} \frac{\sharp(S_{\{j\}} \cap [0,n])}{n+1} + \limsup_{n\to n} \frac{\sharp(S_{\{-j\}} \cap [0,n])}{n+1}.$$

Define $N_0 = \min S_{\{j\}}$. Observe that $\frac{\sharp(S_{\{j\}}\cap[0,N_0])}{N_0+1} = \frac{1}{N_0+1}$. Let $N_1 = \min S_{\{j\}} \setminus \{N_0\}$. We have that $\frac{\sharp(S_{\{j\}}\cap[0,N_1])}{N_1+1} = \frac{2}{N_1+1} < \frac{2}{N_0+2(2^{m+3}-1)+1}$. Following this construction, for every $r \geq 1$, we define $N_r = \min(S_{\{j\}} \setminus \{N_l\}_{l=0}^{r-1})$. Observe that

$$\begin{aligned} \tfrac{\sharp(S_{\{j\}}\cap[0,N_r])}{N_r+1} &= \tfrac{r+1}{N_r+1} \\ &< \tfrac{r+1}{r(2^{m+4}-2+\frac{1}{r})}. \end{aligned}$$

Since

$$\lim_{r\to\infty} \frac{r+1}{r} \frac{1}{2^{m+4}-2+\frac{1}{r}} = \frac{1}{2^{m+4}-2},$$

then

$$\lim_{r\to\infty} \frac{\sharp(S_{\{j\}} \cap [0,N_r])}{N_r+1} \leq \frac{1}{2(2^{m+3}-1)} < \frac{1}{2^{m+3}}.$$

Similarly, we obtain

$$\lim_{r\to\infty} \frac{\sharp(S_{\{-j\}} \cap [0, N_r])}{N_r + 1} < \frac{1}{2^{m+3}}.$$

Thus,

$$\lim_{r\to\infty} \frac{\sharp(S_{\{-j,j\}} \cap [0, N_r])}{N_r + 1} \leq \frac{1}{2^{m+2}}.$$

Part 2: By Lemma 3.11 and Part 1, we have that for all $j \in \mathbb{Z}$ with $x_j = \square$, and

$$-(m + 2 + \sum_{l=0}^{m+2} 2^l) \leq j \leq m + 2 + \sum_{l=0}^{m+2} 2^l,$$

then

$$3\overline{D}(S_{\{-d,d\}}) \geq \overline{D}(S_{\{-j,j\}}),$$

where $d = m + 3 + \sum_{l=0}^{m+2} 2^l$. Since $\overline{D}(S_{\{-d,d\}}) \leq \frac{1}{3}\frac{1}{2^{m+2}}$, then $\overline{D}(S_{\{-j,j\}}) \leq \frac{1}{2^{m+2}}$. Therefore, Proposition 2.20 gives us that x is a mean equicontinuity point.

$\square$

In the following statement, we state that the set of equicontinuity points is dense. The proof of Lemma 3.13 is very similar to the proof of Proposition 3.12.

Lemma 3.13. *Let $m > 0$, $w \in A^m$ and*

$$x := \cdots \boxbar\,\square^{2^2}\boxbar\,\square^{2^1}\boxbar\,\square^{2^0}.w\boxbar\,\square^{2^0}\boxbar\,\square^{2^1}\boxbar\,\square^{2^2}\cdots.$$

We have that x is a mean equicontinuity point.

We now present the main theorem of this section which states that the set of almost equicontinuous CA is properly contained in the set of almost mean equicontinuous CA.

Theorem 3.14. *$(A^{\mathbb{Z}}, T)$ has no equicontinuity points (hence is not almost equicontinuous). However, it is almost mean equicontinuous.*

Proof. The first statement follows immediately from Proposition 3.9.

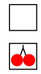
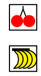
Now, let $x \in A^{\mathbb{Z}}$, $m \geq 0$, and $w = x_{[0,m]}$. We set

$$y := \cdots \boxbar\,\square^{2^2}\boxbar\,\square^{2^1}\boxbar\,\square^{2^0}.w\boxbar\,\square^{2^0}\boxbar\,\square^{2^1}\boxbar\,\square^{2^2}\cdots.$$

From Lemma 3.13, we conclude that y is a mean equicontinuity point. Therefore, $(A^{\mathbb{Z}}, T)$ is almost mean equicontinuous. $\square$

3.2 The Pacman level 2 CA

Let $A = \{\square, \boxbar,$ , , , $\}$, $A_2 = \{\square,$ , $\}$ and $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ the Pacman CA of Section 3.1. We define $T_2 : A_2^{\mathbb{Z}} \to A_2^{\mathbb{Z}}$ as

$T_2 x_i =$
- $\square$ *if* $x_i = \square$;
-  *if* $x_i =$ ;
-  *if* $x_i =$ .

Now we will define some sort of skew product (see Example **??**). We define $A_P := A \times A_2$, and the map $T_P : A_P^{\mathbb{Z}} \to A_P^{\mathbb{Z}}$ as

$$T_P x_i \quad = \quad \begin{cases} (Tx_i, T_2 x_i) & if \quad x_i \notin \{(\text{👾}, \text{🍒}), (\text{👻}, \text{🍒}), (\text{😨}, \text{🍒})\}; \\ (Tx_i, \text{🍒}) & if \quad x_i \in \{(\text{👾}, \text{🍒}), (\text{👻}, \text{🍒}), (\text{😨}, \text{🍒})\}. \end{cases}$$

The following result is almost identical to Lemma 3.5; in fact, its proof follows immediately from Lemma 3.5.

Lemma 3.15. *Let $m > 0$, $w \in A_P^m$, and*

$$x = {}^{\infty}(\square, \square).w(\boxed{\|}, \square)(\square, \square)^{\infty}.$$

Then, there exists $N > 0$ such that for all $n \geq N$ and all $0 \leq i \leq |w|$,

$$T_P^n x_i \in \{(p, q) : p \in \{\square, \boxed{\|}\} \wedge q \in A_2\}.$$

We want to show that $(A_P^{\mathbb{Z}}, T_P)$ is not almost mean equicontinuous. Using Proposition 1.12, we need to find a non-empty open set that does not contain any mean equicontinuity points.

Lemma 3.16. *Let $m > 0$ and $w \in A_P^m$ such that $w_0 = (\boxed{\|}, \text{🍒})$. Then, there exist $x, y \in A_P^{\mathbb{Z}}$ such that*

$$x_{[0,|w|-1]} = y_{[0,|w|-1]} = w$$

and the set

$$\mathbb{Z}_{n \geq 0} \setminus \{n \in \mathbb{Z}_{n \geq 0} : T_P^n x_0 \neq T_P^n y_0\}$$

is finite.

Proof. Let $w \in A_P^m$ as in the hypothesis of the statement. Let us define

$$x := {}^{\infty}(\square, \square).w(\boxed{\|}, \square)(\text{👾}, \square)(\square, \square)^{\infty}$$

and

$$y := {}^{\infty}(\square, \square).w(\boxed{\|}, \square)(\square, \square)^{\infty}.$$

Using Lemma 3.15, we can assume, without loss of generality, that $w_i \in \{(p, q) : p \in \{\square, \boxed{\|}\} \wedge q \in A_2\}$. Now, there exists $N > 0$ such that $T_P^N x_0 = (\text{👻}, q)$, where $q \in \{\text{🍒}, \text{🍌}\}$. Meanwhile, for all $i \geq 0$, we have that $T_P^i y_0 = (\boxed{\|}, q)$ with $q \in \{\text{🍒}, \text{🍌}\}$. We have two cases to prove.

Case 1: $T_P^N x_0 = (\text{👻}, \text{🍒})$.

This implies that $T_P^{N+1} x_0 = (\boxed{\|}, \text{🍒})$. Meanwhile, $T_P^{N+1} y_0 = (\boxed{\|}, \text{🍌})$. Therefore, we can easily see that $T_P^{N+i} x_0 \neq T_P^{N+i} y_0$, for all $i > 0$.

Case 2: $T_P^N x = (\text{👻}, \text{🍌})$.

Again, we have that $T_P^{N+1} x_0 = (\boxed{\|}, \text{🍒})$. So, $T_P^{N+i} x_0 = T_P^{N+i} y_0$ for all $i \geq 0$. In this case, we redefine

$$x := {}^{\infty}(\square, \square).w(\boxed{\|}, \square)(\square, \square)(\text{👾}, \square)(\square, \square)^{\infty}$$

and we finish the proof using a similar argument as the one given in Case 1.

$\square$

The following statement follows immediately from Lemma 3.16.

Lemma 3.17. *Let $x \in A_P^{\mathbb{Z}}$ such that $x_0 = (\ ,\ )$. Then, x is not a mean equicontinuity point.*

Notice that, for all $\varepsilon > 0$, any $y \in B_\varepsilon(x)$, where $x_0 = (\ ,\ )$, is not a mean equicontinuity point.

Theorem 3.18. *$(A_P^{\mathbb{Z}}, T_P)$ is neither mean sensitive nor almost mean equicontinuous.*

Proof. Let us show that $(A_P^{\mathbb{Z}}, T_P)$ is not mean sensitive, i.e., for every $\varepsilon > 0$ there exists a open set $U \subset A_P^{\mathbb{Z}}$ such that for every $x, y \in U$

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T_P^i x, T_P^i y)}{n+1} < \varepsilon.$$

From Proposition 3.13,

$$x := \cdots (\ ,\ )(\ ,\ )^{2^1}(\ ,\ )(\ ,\ )^{2^0}.(\ ,\ )(\ ,\ )^{2^0}(\ ,\ )(\ ,\ )^{2^1}\cdots$$

is a mean equicontinuity point. From Proposition 1.12, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y, z \in B_\delta(x)$

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T_P^i y, T_P^i z)}{n+1} < \varepsilon.$$

Therefore, $(A_P^{\mathbb{Z}}, T_P)$ is not mean sensitive.

The fact that $(A_P^{\mathbb{Z}}, T_P)$ is not almost mean equicontinuous follows immediately from Lemma 3.17. $\square$

Naturally, this example cannot be a transitive CA.

3.3 Almost diam-mean equicontinuity 1

All the results in this section and their proofs are contained in [4].

The strongest form of diameter sensitivity is called cofinitely sensitivity (see Definition 1.19)[25]. It is easy to see that every cofinitely sensitive TDS is diam-mean sensitive.

In the proof of Lemma 3.3, we know exactly when a ghost crosses two doors in N iterations. So, it is clear that, if we move the ghost one place to the right it will take $N+1$ iterations. To generalize this idea we show the following:

Lemma 3.19. *Let (X, T) be the Pacman CA and $w \in A^+$ a finite word. We define the points*

$$x^i = {}^{\infty}\ .w\ \ ^{i}\ \ ^{\infty}.$$

Then, there exist $N, M \geq 0$ such that for every $j \in \mathbb{Z}_{\geq 0}$ we have that $T^{N+j} x_0^{M+j} \in \{\ ,\ ,\ \}$.

Proof. First assume that $w \in \{\ ,\ \}^+$. By simple application of the rules of the Pacman CA we can conclude that there exists $N_0 \geq 0$ such that $T^{N_0} x_0^1 \in \{\ ,\ ,\ \}$. Furthermore, since $Tx^i = x^{i-1}$ we obtain the result for $M = 0$; that is, for every $j \in \mathbb{Z}_{\geq 0}$ we have that $T^{N_0+j} x_0^j \in \{\ ,\ ,\ \}$.

For the general case, let $w \in A^m$ and $x = {}^{\infty}\square.w\boxed{\square}\,\square^{\infty}$. Lemma 3.5 implies that there exists $M' > 0$ such that

$$T^{M'}x_i \in \{\square, \boxed{\square}\}\ \forall\ i \geq 0.$$

Since points $y^i := T^{M'}x^{M'+i}$ look exactly like the cases presented on the first part of the proof, we conclude that there exist $N, M \geq 0$ such that for every $j \in \mathbb{Z}_{\geq 0}$ we have that $T^{N+j}x_0^{M+j} \in \{$, , $\}$.

$\square$

With the help of Lemma 3.19, we can construct a sequence of elements in any open set that will help us to show that the Pacman CA is cofinitely sensitive.

Proposition 3.20. *The Pacman CA is cofinitely sensitive.*

Proof. Let w be a finite word and let $x \in [w]_0$ such that $x = {}^{\infty}\square.w\boxed{\square}\,\square^{\infty}$. Let $(y^i)_{i=0}^{\infty} \subset [w]_0$ such that

$y^i = {}^{\infty}\square.w\boxed{\square}\,\square^{i}$$\square^{\infty}$.

By Lemma 3.5, there exists $M \geq 0$ such that for all $n \geq M$ we have that

$$T^n \in \{\square, \boxed{\square}\}\ \forall\ i \geq 0.$$

Hence, by Lemma 3.19, there exists $N \geq 0$ such that

$$d(T^{N+j}x, T^{N+j}y^{M+j}) = 1$$

for all $j \geq 0$. Therefore, the Pacman CA is cofinitely sensitive. $\square$

Using a similar strategy, it can be shown that the Pacman Level 2 CA is also cofinitely sensitive.

3.4 Shift-odometer example

The results of this section appeared in [5].

In this section, we will construct an almost diam-mean equicontinuous CA that is not almost equicontinuous.

First, we define a CA that resembles an odometer (compare with Example 1.6). Let $A_1 = \{\square\} \cup \mathbb{Z}_3 = \{\square, \boxed{0}, \boxed{1}, \boxed{2}\}$. We define the CA $T_1 : A_1^{\mathbb{Z}} \to A_1^{\mathbb{Z}}$ locally as follows: $T_1 x_i = \square$ if and only if $x_i = \square$, otherwise $T_1 x_i \in \{x_i, (x_i + 1)\text{mod}3\}$, with $T_1 x_i = (x_i + 1)\text{mod}3$ if and only if $x_{i+1} \in \{\square, \boxed{2}\}$. In other words:

$$T_1 x_i = \begin{cases} \square & if \quad x_i = \square \\ \boxed{0} & if \quad (x_i = \boxed{2} \wedge x_{i+1} \in \{\square, \boxed{2}\}) \vee \\ & \qquad (x_i = \boxed{0} \wedge x_{i+1} \in A_1 \setminus \{\square, \boxed{2}\}); \\ \boxed{1} & if \quad (x_i = \boxed{0} \wedge x_{i+1} \in \{\square, \boxed{2}\}) \vee \\ & \qquad (x_i = \boxed{1} \wedge x_{i+1} \in A_1 \setminus \{\square, \boxed{2}\}); \\ \boxed{2} & if \quad x_i = \boxed{1} \wedge x_{i+1} \in \{\square, \boxed{2}\}) \vee \\ & \qquad (x_i = \boxed{2} \wedge x_{i+1} \in A_1 \setminus \{\square, \boxed{2}\}). \end{cases}$$

The following example will help us to understand $(A_1^{\mathbb{Z}}, T_1)$. Also, it helps us to see why $(A_1^{\mathbb{Z}}, T_1)$, to some extent, resembles an odometer (see Example 1.6).

Example 3.21. Let $x \in A_1^{\mathbb{Z}}$ such that $x_{[0,5]} = \square\boxed{0}\,\boxed{0}\,\boxed{0}\,\boxed{0}\square$. We have that

$$\begin{array}{lcllllll@{\qquad}lcllllll}
T_1^{0}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \square & T_1^{14}x_{[0,5]} & = & \square & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{2} & \square \\
T_1^{1}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \square & T_1^{15}x_{[0,5]} & = & \square & \boxed{1} & \boxed{0} & \boxed{2} & \boxed{0} & \square \\
T_1^{2}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{2} & \square & T_1^{16}x_{[0,5]} & = & \square & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{1} & \square \\
T_1^{3}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & \square & T_1^{17}x_{[0,5]} & = & \square & \boxed{1} & \boxed{2} & \boxed{2} & \boxed{2} & \square \\
T_1^{4}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & \square & T_1^{18}x_{[0,5]} & = & \square & \boxed{2} & \boxed{0} & \boxed{0} & \boxed{0} & \square \\
T_1^{5}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{2} & \square & T_1^{19}x_{[0,5]} & = & \square & \boxed{2} & \boxed{0} & \boxed{0} & \boxed{1} & \square \\
T_1^{6}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{2} & \boxed{0} & \square & T_1^{20}x_{[0,5]} & = & \square & \boxed{2} & \boxed{0} & \boxed{0} & \boxed{2} & \square \\
T_1^{7}x_{[0,5]} & = & \square & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{1} & \square & T_1^{21}x_{[0,5]} & = & \square & \boxed{2} & \boxed{0} & \boxed{1} & \boxed{0} & \square \\
T_1^{8}x_{[0,5]} & = & \square & \boxed{0} & \boxed{2} & \boxed{2} & \boxed{2} & \square & T_1^{22}x_{[0,5]} & = & \square & \boxed{2} & \boxed{0} & \boxed{1} & \boxed{1} & \square \\
T_1^{9}x_{[0,5]} & = & \square & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & \square & T_1^{23}x_{[0,5]} & = & \square & \boxed{2} & \boxed{0} & \boxed{1} & \boxed{2} & \square \\
T_1^{10}x_{[0,5]} & = & \square & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \square & T_1^{24}x_{[0,5]} & = & \square & \boxed{2} & \boxed{0} & \boxed{2} & \boxed{0} & \square \\
T_1^{11}x_{[0,5]} & = & \square & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{2} & \square & T_1^{25}x_{[0,5]} & = & \square & \boxed{2} & \boxed{1} & \boxed{2} & \boxed{1} & \square \\
T_1^{12}x_{[0,5]} & = & \square & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} & \square & T_1^{26}x_{[0,5]} & = & \square & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} & \square \\
T_1^{13}x_{[0,5]} & = & \square & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{1} & \square & T_1^{27}x_{[0,5]} & = & \square & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \square.
\end{array}$$

Now that we have looked at Example 3.31, we can see that "empty" remains fixed and is not influenced by any of its neighbors; a difference from the odometer. In addition, the following remarks add observations that are easy to verify.

Remark 3.22. Note that the CA T_1 has anticipation 1 and memory 0; that is, T_1x_0 only depends on x_0 and x_1. Hence, if $x_i = y_{i+N}$ for every $i \le 1$ then $T_1x_i = T_1y_{i+N}$ for every $i \le 0$.

Remark 3.23. It is not difficult to see that T_1 is almost equicontinuous. In fact, $\square$ is a **blocking word**, i.e., if $x, y \in A_1^{\mathbb{Z}}$ with $x_{[m',m]} = y_{[m',m]}$ and $x_m = \square$, then $T_1^n x_{[m',m]} = T_1^n y_{[m',m]}$ for every $n > 0$. Moreover, in this situation, one can check that there exist $M > 0$ and $p > 0$ such that $T_1^{M+ip}x_{[m',m]} = T_1^M x_{[m',m]}$ for all $i \ge 0$.

Given a CA (X, T), we say a point $x \in X$ is **periodic** with period p if $T^p x = x$. This should not be confused with the statement, $T^n x_0$ is periodic, which means that the sequence with respect to n is periodic. We will now state some statements that help us to understand the behavior of T_1.

Lemma 3.24. *Let* $x := {}^{\infty}\square.\boxed{0}\,\square^{\infty}$. *Then* x *is a periodic point with period* 3.

Proof. Observe that $T_1x_i = \square$ if and only if $x_i = \square$. Hence,

- $T_1x_0 = \boxed{1}$ and $T_1x_i = \square$ for all $i \in \mathbb{Z} \setminus \{0\}$;
- $T_1^2x_0 = \boxed{2}$ and $T_1^2x_i = \square$ for all $i \in \mathbb{Z} \setminus \{0\}$; and
- $T_1^3x_0 = \boxed{0}$ and $T_1^3x_i = \square$ for all $i \in \mathbb{Z} \setminus \{0\}$.

Therefore, x is a periodic point with period 3. $\square$

The following lemma shows that, if we start adding $\boxed{0}$ to x we can change its periodicity; although this does not necessarily have to happen as we will see in Lemma 3.27.

Lemma 3.25. *Let* $x := {}^{\infty}\boxed{}.\boxed{0}\,\boxed{0}\,\boxed{}^{\infty}$. *Then* x *is a periodic point with period* 9.

Proof. Remark 3.23 and Lemma 3.24 implies that $T_1^{3k}x_1 = 1$ and $T_1^{3k}x_i = \boxed{}$ for all $i \in \mathbb{Z} \setminus \{0, 1\}$ and for all $k \in \mathbb{Z}_{\geq 0}$. Thus, we have that:

- $T_1^3 x_{[0,1]} = \boxed{1}\ \boxed{0}$ and $T_1^3 x_i = \boxed{}$ for all $i \in \mathbb{Z} \setminus \{0, 1\}$;
- $T_1^6 x_{[0,1]} = \boxed{2}\ \boxed{0}$ and $T_1^6 x_i = \boxed{}$ for all $i \in \mathbb{Z} \setminus \{0, 1\}$;
- $T_1^9 x_{[0,1]} = \boxed{0}\ \boxed{0}$ and $T_1^9 x_i = \boxed{}$ for all $i \in \mathbb{Z} \setminus \{0, 1\}$. Therefore, x is a periodic point with period 9.

$\square$

From the proof of Lemma 3.25, we can conclude the next result. Although the following lemma seems redundant because it is very similar to Lemma 3.25, it helps us to see what exactly happens at the $0th$ coordinate of x.

Lemma 3.26. *If* $x := {}^{\infty}\boxed{}.\boxed{0}\,\boxed{0}\,\boxed{}^{\infty}$, *then*

1. $T_1^i x_0 = \boxed{0}$ *for all* $0 \leq i \leq 2$;
2. $T_1^i x_0 = \boxed{1}$ *for all* $3 \leq i \leq 5$;
3. $T_1^i x_0 = \boxed{2}$ *for all* $6 \leq i \leq 8$.

As we mentioned before, something to be careful about is that the period does not necessarily increase if the amount of $\boxed{0}$s increases (one of the differences with an odometer). The following result is an evidence of the previous comment.

Lemma 3.27. *Let* $y := {}^{\infty}\boxed{}.\boxed{0}\,\boxed{0}\,\boxed{0}\,\boxed{}^{\infty}$. *Then* x *and* y *are periodic points with period* 9.

Proof. By Lemmas 3.25 and 3.26 we have that

$$T_1^i y_0 = \boxed{0},$$

for all $0 \leq i \leq 6$. Observe that

$$T_1^{6+i} y_0 = T_1^i y_2,$$

for all $0 \leq i \leq 3$. Hence, $T_1^9 y_{[0,2]} = \boxed{0}\ \boxed{0}\ \boxed{0}$ and $T_1^9 y_i = \boxed{}$ for all $i \in \mathbb{Z} \setminus \{0, 1, 2\}$. $\square$

Since the periodicity of x changes if it has one or two $\boxed{0}$'s while the periodicity is the same when it has two or three $\boxed{0}$'s, it would be nice to determine exactly how many $\boxed{0}$'s it takes to change the periodicity and when it is preserved. In other words, we want to generalize Lemmas 3.24 to 3.27. For that endeavor, we need the following result, which gives us some conditions for when we want the first n iterations, of two close points of $(A_1^{\mathbb{Z}}, T_1)$, to stay close.

Lemma 3.28. *Let* $m, k > 0$, *and* $x, y \in A_1^{\mathbb{Z}}$ *such that* $x_{[0,k]} = y_{[0,k]}$, *and* $\{T_1^i x_{k+1}, T_1^i y_{k+1}\} \subset \{\boxed{}, \boxed{2}\}$ *for every* $i \in [0, m]$. *Then* $T_1^i x_{[0,k]} = T_1^i y_{[0,k]}$ *for every* $i \in [0, m]$.

Proof. The proof can be obtained using Remark 3.22, the shift commuting property of a CA, and the fact that if x', y' satisfies that $x'_0 = y'_0$ and $\{x'_1, y'_1\} \subset \{\boxed{}, \boxed{2}\}$, then $T_1 x'_0 = T_1 y'_0$.

$\square$

The following proposition gives the desired generalization of Lemmas 3.24 to 3.27.

Proposition 3.29. *Let* $l \geq 1$, $1 \leq j < 2^l$, $x^l := {}^{\infty}\square.\boxed{0}^{2^l}\square^{\infty}$ *and* $y^{l,j} := {}^{\infty}\square.\boxed{0}^{2^l+j}\square^{\infty}$. *Then:*

1. x^l *and* $y^{l,j}$ *are periodic points with period* 3^{l+1}.
2. *We have that:*
   - $T_1^i x_0^l = \boxed{0}$ *for all* $0 \leq i < 3^l$;
   - $T_1^i x_0^l = \boxed{1}$ *for all* $3^l \leq i < 2(3^l)$;
   - $T_1^i x_0^l = \boxed{2}$ *for all* $2(3^l) \leq i < 3^{l+1}$.

Proof. Let us shows part 2 firstly, we will prove the result by induction on l. From Lemma 3.27, we obtain the result for $l = 1$. Assume the result holds for $l = k$. Now, let $l = k + 1$. By the induction hypothesis and Remark 3.23 we have:

- $T_1^i x_{2^k}^{k+1} = \boxed{0}$ for all $0 \leq i < 3^k$;
- $T_1^i x_{2^k}^{k+1} = \boxed{1}$ for all $3^k \leq i < 2(3^k)$;
- $T_1^i x_{2^k}^{k+1} = \boxed{2}$ for all $2(3^k) \leq i < 3^{k+1}$.

Hence, we have that $T_1^i x_{[0,2^k)}^{k+1} = \boxed{0}^{2^k}$ for all $0 \leq i \leq 2(3^k)$. Observe that

$$T_1^{2(3^k)} x_{[0,2^k]}^{k+1} = \boxed{0}^{2^k}\boxed{2} \text{ and } x_{[2^k,2^{k+1}]}^{k+1} = \boxed{0}^{2^k}\square.$$

Using Lemma 3.28 and the fact that T_1 commutes with the shift, we obtain that

$$T_1^{2(3^k)+i} x_{[0,2^k)}^{k+1} = T_1^i x_{[2^k,2^{k+1})}^{k+1},$$

for all $0 \leq i < 3^k$. This implies that $T_1^i x_0^{k+1} = \boxed{0}$ for all $0 \leq i < 3^{k+1}$.

By the induction hypothesis we have that $y^{k,2^k-1} = {}^{\infty}\square\,.\boxed{0}^{2^{k+1}-1}\square^{\infty}$ has period 3^{k+1}. Since $x_{(0,2^{k+1}]}^{k+1} = y_{[0,2^{k+1}-1]}^{k,2^k-1}$, Remark 3.23 gives us that

$$T_1^{3^{k+1}} x_{(0,2^{k+1}]}^{k+1} = x_{(0,2^{k+1}]}^{k+1} = \boxed{0}^{2^{k+1}-1}\square.$$

Thus,

$$T_1^{3^{k+1}-1} x_{[0,2^{k+1}]}^{k+1} = \boxed{0}\,\boxed{2}^{2^{k+1}-1}\square, \text{ and}$$

$$T_1^{3^{k+1}} x_{[0,2^{k+1}]}^{k+1} = \boxed{1}\,\boxed{0}^{2^{k+1}-1}\square.$$

By this, and a similar use of the induction hypothesis, we obtain that

- $T_1^{3^{k+1}+i} x_{2^k}^{k+1} = \boxed{0}$ for all $0 \leq i < 3^k$;
- $T_1^{3^{k+1}+i} x_{2^k}^{k+1} = \boxed{1}$ for all $3^k \leq i < 2(3^k)$;

- $T_1^{3^{k+1}+i} x_{2^k}^{k+1} = \boxed{2}$ for all $2(3^k) \leq i < 3^{k+1}$.

Then, $T_1^{3^{k+1}+i} x_{[0,2^k)}^{k+1} = \boxed{1}\ \boxed{0}^{2^k-1}$ for all $0 \leq i \leq 2(3^k)$. So, using Lemma 3.28 and the fact that T_1 commutes with the shift, we have that

$$T_1^{3^{k+1}+i} x_{[0,2^k)}^{k+1} = T_1^i x_{[2^k,2^{k+1})}^{k+1}$$

for all $2(3^k) \leq i < 3^{k+1}$. Therefore, $T_1^i x_0^{k+1} = \boxed{1}$ for all $3^{k+1} \leq i < 2(3^{k+1})$. In a similar way, we have that $T_1^i x_0^{k+1} = \boxed{2}$ for all $2(3^{k+1}) \leq i < 3^{k+2}$. Hence,

- $T_1^i x_0^{k+1} = \boxed{0}$ for all $0 \leq i < 3^{k+1}$;
- $T_1^i x_0^{k+1} = \boxed{1}$ for all $3^{k+1} \leq i < 2(3^{k+1})$; and
- $T_1^i x_0^{k+1} = \boxed{2}$ for all $2(3^{k+1}) \leq i < 3^{k+2}$.

Therefore, x^{k+1} has period $p = 3^{k+2}$. With this, we conclude part 2 of the proposition.

Now, let $0 \leq j < 2^{k+1}$. Using part 2 of the proposition, we have that $T_1^i y_{[0,j)}^{k+1,j} = \boxed{0}^j$ for all $0 \leq i \leq 2(3^{k+1})$. Using Lemma 3.28 and the fact that T_1 commutes with the shift, we obtain that

$$T_1^{2(3^{k+1})+i} y_{[0,j)}^{k+1,j} = T_1^i x_{[2^{k+1}-j,2^{k+1})}^k$$

for all $0 \leq i \leq 3^{k+1}$. Since x^k has period 3^{k+1} we have that

$$T_1^{3^{k+1}} x_{[2^{k+1}-j,2^{k+1}]}^{k+1} = \boxed{0}^j \boxed{}.$$

Therefore, $y^{k+1,j}$ has period $2(3^{k+1}) + 3^{k+1} = 3^{k+2}$. With this we conclude part 1. $\square$

Already at this point in the section, we could easily think of the elements of $A_1^{\mathbb{Z}}$ that remain fixed or eventually fixed. The following proposition tells us when the elements of $A_1^{\mathbb{Z}}$ do not remain fixed or eventually fixed.

Proposition 3.30. *If $x \in A_1^{\mathbb{Z}}$ is such that $x_j = \boxed{}$ for some $j \in \mathbb{Z}$, then for all $i \in \mathbb{N}$ we have $T_1^n x_{j-i} \in \{\boxed{}, \boxed{2}\}$ for infinitely many $n > 0$.*

Proof. We will prove this result by induction on i. The result follows for $i = 0$ using straightforward applications of the rules of the automaton (as in Lemma 3.24). Assume the result holds for i. We may assume that $x_{j-i} \neq \boxed{}$ and $x_{j-i-1} \neq \boxed{}$; otherwise, the result is straightforward. Hence, we have that $\{n > 0 : T_1^n x_{j-i} = \boxed{2}\}$ is infinite. Furthermore, by the rules of the automaton for every $n' \in \{n > 0 : T_1^n x_{j-i} = \boxed{2}\}$ we have that $T^{n'+1} x_{j-i-1} = (T^{n'} x_{j-i-1} + 1) \mathrm{mod} 3$. With this we can conclude that for all $i \in \mathbb{N}$ we have $T_1^n x_{j-i} \in \{\boxed{}, \boxed{2}\}$ for infinitely many $n > 0$. $\square$

Now we will combine T_1 with the shift map. Let $A_2 = \{\boxed{}, \boxed{\leftarrow}\}$, and $\sigma : A_2^{\mathbb{Z}} \to A_2^{\mathbb{Z}}$ the shift map.

Let $A = A_1 \times A_2$. At times we will identify A with the following set

$$A = \{\boxed{}, \boxed{0}, \boxed{1}, \boxed{2}, \boxed{\leftarrow}, \boxed{\overleftarrow{0}}, \boxed{\overleftarrow{1}}, \boxed{\overleftarrow{2}}\}.$$

Note that with this notation, we identify the point $\boxed{} \times \boxed{} \in A$ with $\boxed{}$. In general it will be clear which $\boxed{}$ we are referring to. Let $\gamma_1 : A \to A_1$ and $\gamma_2 : A \to A_2$ be the projection functions. We also extend such functions to $A^{\mathbb{Z}} \to A_1^{\mathbb{Z}}$ and $A^{\mathbb{Z}} \to A_2^{\mathbb{Z}}$, respectively.

We define the CA $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ locally (and coordinate-wise) as the only CA satisfying:

$$
\begin{aligned}
(\gamma_1(Tx))_i &= (T_1\gamma_1(x))_i, \text{ and} \\
(\gamma_2(Tx))_i &= \begin{cases} (\sigma\gamma_2(x))_i & \text{if } \left\{\boxed{\leftarrow}, \boxed{\overleftarrow{2}}\right\} \cap \{x_i, x_{i+1}\} \neq \emptyset, \\ (\gamma_2(x))_i & \qquad\qquad \text{otherwise.} \end{cases}
\end{aligned}
$$

In other words, on the first coordinate T acts exactly as T_1; on the second coordinate, an arrow advances to the left if and only if the first coordinate is a $\boxed{}$ or a $\boxed{2}$. In case two arrows overlap they superimpose each other.

In the Introduction, we mentioned that the previous construction is a "local" skew product. On a skew product the phase space is the product, one coordinate acts normally and the second acts only if the first coordinate is in a certain position. The main difference is that here the shift only acts locally.

Though the next table is not needed for the proofs, we provide it in case it is of assistance to the reader.

$$
Tx_i \quad = \quad \begin{cases}
\boxed{} & if \quad x_i \in \{\boxed{}, \boxed{\leftarrow}\} \wedge x_{i+1} \in A \setminus \{\boxed{\leftarrow}, \boxed{\overleftarrow{2}}\}; \\
\boxed{\leftarrow} & if \quad x_i \in \{\boxed{}, \boxed{\leftarrow}\} \wedge x_{i+1} \in \{\boxed{\leftarrow}, \boxed{\overleftarrow{2}}\}; \\
\boxed{0} & if \quad (x_i \in \{\boxed{2}, \boxed{\overleftarrow{2}}\} \wedge x_{i+1} \in \{\boxed{}, \boxed{2}\})\vee; \\
 & \qquad (x_i = \boxed{0} \wedge x_{i+1} \in A \setminus \{\boxed{}, \boxed{\leftarrow}, \boxed{2}, \boxed{\overleftarrow{2}}\}); \\
\boxed{1} & if \quad (x_i = \boxed{0} \wedge x_{i+1} \in \{\boxed{}, \boxed{2}\})\vee; \\
 & \qquad (x_i = \boxed{1} \wedge x_{i+1} \in A \setminus \{\boxed{}, \boxed{\leftarrow}, \boxed{2}, \boxed{\overleftarrow{2}}\}); \\
\boxed{2} & if \quad (x_i = \boxed{1} \wedge x_{i+1} \in \{\boxed{}, \boxed{2}\})\vee \\
 & \qquad (x_i \in \{\boxed{2}, \boxed{\overleftarrow{2}}\} \wedge x_{i+1} \in A \setminus \{\boxed{}, \boxed{\leftarrow}, \boxed{2}, \boxed{\overleftarrow{2}}\}); \\
\boxed{\overleftarrow{0}} & if \quad (x_i \in \{\boxed{2}, \boxed{\overleftarrow{2}}\} \wedge x_{i+1} \in \{\boxed{\leftarrow}, \boxed{\overleftarrow{2}}\})\vee; \\
 & \qquad (x_i = \boxed{\overleftarrow{0}} \wedge x_{i+1} \in A \setminus \{\boxed{}, \boxed{\leftarrow}, \boxed{2}, \boxed{\overleftarrow{2}}\}); \\
\boxed{\overleftarrow{1}} & if \quad (x_i = \boxed{0} \wedge x_{i+1} \in \{\boxed{\leftarrow}, \boxed{\overleftarrow{2}}\})\vee \\
 & \qquad (x_i = \boxed{\overleftarrow{0}} \wedge x_{i+1} \in \{\boxed{}, \boxed{\leftarrow}, \boxed{2}, \boxed{\overleftarrow{2}}\})\vee \\
 & \qquad (x_i = \boxed{\overleftarrow{1}} \wedge x_{i+1} \in A \setminus \{\boxed{}, \boxed{\leftarrow}, \boxed{2}, \boxed{\overleftarrow{2}}\}); \\
\boxed{\overleftarrow{2}} & if \quad (x_i = \boxed{1} \wedge x_{i+1} \in \{\boxed{\leftarrow}, \boxed{\overleftarrow{2}}\})\vee \\
 & \qquad (x_i = \boxed{\overleftarrow{1}} \wedge x_{i+1} \in \{\boxed{}, \boxed{\leftarrow}, \boxed{2}, \boxed{\overleftarrow{2}}\}).
\end{cases}
$$

Example 3.31. Let $x \in A^{\mathbb{Z}}$ such that $x_{[0,6]} = \boxed{}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{\leftarrow}\boxed{}$. We have that

$T^0x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{\leftarrow}$	$\square$	$T^{14}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\square$	$\square$
$T^1x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{\Lsh 1}$	$\square$	$\square$	$T^{15}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\boxed{2}$	$\boxed{0}$	$\square$	$\square$
$T^2x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{\Lsh 2}$	$\square$	$\square$	$T^{16}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{1}$	$\square$	$\square$
$T^3x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\square$	$\square$	$T^{17}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\square$	$\square$
$T^4x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{\Lsh 1}$	$\boxed{1}$	$\square$	$\square$	$T^{18}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 2}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\square$	$\square$
$T^5x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{\Lsh 1}$	$\boxed{2}$	$\square$	$\square$	$T^{19}x_{[0,6]} =$	$\boxed{\leftarrow}$	$\boxed{2}$	$\boxed{0}$	$\boxed{0}$	$\boxed{1}$	$\square$	$\square$
$T^6x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{\Lsh 2}$	$\boxed{0}$	$\square$	$\square$	$T^{20}x_{[0,6]} =$	$\square$	$\boxed{2}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	$\square$	$\square$
$T^7x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{\Lsh 1}$	$\boxed{2}$	$\boxed{1}$	$\square$	$\square$	$T^{21}x_{[0,6]} =$	$\square$	$\boxed{2}$	$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\square$	$\square$
$T^8x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{\Lsh 2}$	$\boxed{2}$	$\boxed{2}$	$\square$	$\square$	$T^{22}x_{[0,6]} =$	$\square$	$\boxed{2}$	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\square$	$\square$
$T^9x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\square$	$\square$	$T^{23}x_{[0,6]} =$	$\square$	$\boxed{2}$	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\square$	$\square$
$T^{10}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{1}$	$\square$	$\square$	$T^{24}x_{[0,6]} =$	$\square$	$\boxed{2}$	$\boxed{0}$	$\boxed{2}$	$\boxed{0}$	$\square$	$\square$
$T^{11}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\boxed{0}$	$\boxed{2}$	$\square$	$\square$	$T^{25}x_{[0,6]} =$	$\square$	$\boxed{2}$	$\boxed{1}$	$\boxed{2}$	$\boxed{1}$	$\square$	$\square$
$T^{12}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\boxed{1}$	$\boxed{0}$	$\square$	$\square$	$T^{26}x_{[0,6]} =$	$\square$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\square$	$\square$
$T^{13}x_{[0,6]} =$	$\square$	$\boxed{\Lsh 1}$	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$	$\square$	$\square$	$T^{27}x_{[0,6]} =$	$\square$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\square$	$\square$.

In the Pacman CA we looked at how long it would take for a ghost to go through two doors, but we don't know exactly what iterations the ghost goes through. In the constructed CA, we are able to know in which possible iterations the arrow can pass through consecutive $\boxed{0}$s and exactly in which iterations it does not. Let us start with one $\boxed{0}$.

Lemma 3.32. *Let $x \in A^{\mathbb{Z}}$ such that $x_{[0,2]} = \square\ \boxed{0}\ \square$. Then $T^n x_0 = \square$ for all $n \neq 3k$, where $k \geq 1$.*

Proof. From Lemma 3.24 we have that $\gamma_1(T^n x_1) = \boxed{2}$ if and only if $n = 3k-1$, where $k \geq 1$. Then, $T^n x_0 = \square$ for all $n \neq 3k$, where $k \geq 1$. $\square$

As we saw in the proof of Lemma 3.32, the periodicity of $(T^i x_{[0,2]})_{i \in \mathbb{Z}_{\geq 0}}$ was used for when it is a single $\boxed{0}$. Now, for what we will continue taking advantage of the periodicity but for 2^l $\boxed{0}$'s, where $l \geq 0$.

Lemma 3.33. *Let $l \geq 0$, $x \in A^{\mathbb{Z}}$ such that $x_{[0,2^l+1]} = \square\ \boxed{0}^{2^l} \square$, and $k \in \mathbb{N}$. If $n \neq k(3^{l+1}) + 2(3^l) + 1$ then $T^n x_0 = \square$.*

Proof. By Proposition 3.29 we have that $T^i x_1$ has period 3^{l+1} and

- $\gamma_1(T^i x_1) = \boxed{0}$ for all $0 \leq i < 3^l$;
- $\gamma_1(T^i x_1) = \boxed{1}$ for all $3^l \leq i < 2(3^l)$;
- $\gamma_1(T^i x_1) = \boxed{2}$ for all $2(3^l) \leq i < 3^{l+1}$.

Note that in general for a point $y \in X$, if $\gamma_1(y_1) \notin \{\square, \boxed{2}\}$ then $\gamma_2(Ty_0) = \square$. This implies that $T^i x_0 = \square$ for all $i \leq 2(3^l)$. Furthermore, note that configurations on position 1 imply that $\gamma_1(T_1^i x_2) \neq \boxed{2}$ for all $2(3^l) \leq i < 3^{l+1} - 1$. Hence, $\gamma_2(T^i x_1) = \square$ for all $2(3^l) + 1 \leq i < 3^{l+1}$, and thus $T^i x_0 = \square$ for all $2(3^l) + 2 \leq i \leq 3^{l+1}$. This concludes the proof for $k = 0$. Using periodicity we conclude that $T^n x_0 = \square$ for all $n \neq k(3^{l+1}) + 2(3^l) + 1$, for some $k \in \mathbb{N}$. $\square$

Considering $x \in A^{\mathbb{Z}}$ such that $\gamma_1(x_i) = \boxed{0}$ for all $i \in \mathbb{Z}$ then we have that $\gamma_1(T^n x_i) = \boxed{0}$ for all $i \in \mathbb{Z}$ and for all $n \geq 0$. In other words, the arrows found on x will always stay in the same place. So, Proposition 3.30 guarantees that, at least for a part of the element mentioned in the Proposition 3.30, that the arrows keep moving to the left as we iterate. This motivates the following lemma.

Lemma 3.34. *Let $x \in A^{\mathbb{Z}}$ and $j \in \mathbb{Z}$ such that $\gamma_2(x_i) = \square$ for all $i \geq j$ and $\gamma_1(x_j) = \square$. For every $k \in \mathbb{Z}$ there exists $N > 0$ such that $\gamma_2(T^n x_k) = \square$ for every $n \geq N$.*

Proof. Let us assume that $\gamma_2(x_{j-1}) = \boxed{\leftarrow}$. If $\gamma_1(x_{j-1}) = \square$, then $\gamma_2(Tx_{j-1}) = \square$ and $\gamma_2(Tx_{j-2}) = \boxed{\leftarrow}$. But, if $\gamma_1(x_{j-1}) \in \{\boxed{0}, \boxed{1}, \boxed{2}\}$, then from Proposition 3.30 we have that $\gamma_2(T^n x_{j-1}) = \square$ and $\gamma_2(T^{N+1} x_{j-2}) = \boxed{\leftarrow}$ for $n > N = \min\{n > 0 : \gamma_1(T^n x_{j-1} = \boxed{2})\}$. This case repeats for all $i < j - 1$. $\square$

The following statement will be used to show that $(A^{\mathbb{Z}}, T)$ is not almost equicontinuous.

Proposition 3.35. *Let $m > 0$ and $w \in A^{2m+1}$. There exists $x, y \in A^{\mathbb{Z}}$ with $x_{[-m,m]} = w = y_{[-m,m]}$ such that $T^n x_0 \neq T^n y_0$ for some $n > 0$.*

Proof. We set $x_i = \square$ and $y_i = \boxed{\leftarrow}$, for every $i \in \mathbb{Z}$ such that $|i| > m$. By Lemma 3.34 we have that there exists $N > 0$ such that for all $n \geq N$ we have that $\gamma_2(T^n x_i) = \square$ for all $i \in [-m, m]$. Since $y_i = \boxed{\leftarrow}$ for all $i > m$ and an application of Lemma 3.34 also gives us that $\{n \in \mathbb{N} : \gamma_2(T^n y_0) = \boxed{\leftarrow}\}$ is infinite.

$\square$

Proposition 2.22 helps us to recognize when $x \in A^{\mathbb{Z}}$ is an equicontinuity point by comparing the "density of its columns" with elements close enough to the x. Therefore, it is important to analyze the densities of consecutive $\boxed{0}$s. In the following lemma, we will use the sets $S_J^{DM}(x, m)$ defined in Definition 2.21.

Lemma 3.36. *Let $x \in A^{\mathbb{Z}}$ such that $x_{[0,2^l+1]} = \square\,\boxed{0}^{2^l}\square$, where $l \geq 0$. Then*

$$\overline{D}(S_{\{0\}}^{DM}(x, 2^l + 1)) = \frac{1}{3^{l+1}}.$$

Proof. There exist $y, z \in A^{\mathbb{Z}}$ such that $y_{[0,2^l+1]} = x_{[0,2^l+1]} = z_{[0,2^l+1]}$, $y_{2^l+1+i} = \square = y_{-i}$, and $z_{2^l+1+i} = \boxed{\leftarrow}$ and $z_{-i} = \square$, for all $i \geq 1$. Hence, we have that

$$S_{\{0\}}^{DM}(x, 2^l + 1) = \{i \in \mathbb{N} : i = 3^{l+1}k + 2(3^l) + 1 \ \forall \ k \geq 0\}.$$

Therefore,

$$\begin{aligned}\overline{D}(S_{\{0\}}^{DM}(x, 2^l + 1)) &= \overline{D}(\{i \in \mathbb{N} : i = 3^{l+1}k + 2(3^l) + 1 \ \forall \ k \geq 0\}) \\ &= \limsup_{k \to \infty} \frac{k}{3^{l+1}k + 2(3^l) + 1} = \frac{1}{3^{l+1}}.\end{aligned}$$

$\square$

Remark 3.37. Let $m > 0$ and $x \in A^{\mathbb{Z}}$ so that $\gamma_1(x_m) = \square$ and $\gamma_2(x_i) = \square$ for all $i \in \mathbb{Z}$. Then:

$$S_{\{j\}}^{DM}(x, m) = \{i \in \mathbb{N} : \exists \ y \in B_m(x), \ \gamma_2(T^i y_j) = \boxed{\leftarrow}\}$$

and

$$S_{\{-j\}}^{DM}(x, m) = \{i \in \mathbb{N} : \exists \ y \in B_m(x), \ \gamma_2(T^i y_{-j}) = \boxed{\leftarrow}\},$$

for all $|j| \leq m$.

Lemma 3.38. *Let $l \geq 0$, $w = \square\ \boxed{0}^{2^{l-1}} \square\ \boxed{0}^{2^l} \square$, $x := {}^\infty\square.w\square^\infty$, and $m = 2^l + 2^{l-1} + 2$. Then*

$$\begin{array}{rcl} \overline{D}(S^{DM}_{\{0\}}(x,m)) & = & \overline{D}(S^{DM}_{\{2^{l-1}+1\}}(x,m)), \\ \overline{D}(S^{DM}_{\{-i\}}(x,m)) & = & \overline{D}(S^{DM}_{\{2^{l-1}+1\}}(x,m)), \text{ and} \\ \overline{D}(S^{DM}_{\{i\}}(x,m)) & \leq & (2(3^{l-1})+1)\overline{D}(S^{DM}_{\{2^{l-1}+1\}}(x,m)), \end{array}$$

for all $1 \leq i \leq 2^{l-1}$.

Proof. For any $y \in B_m(x)$ one can check that $\gamma_2(T^n y_{2^{l-1}+1}) = \boxed{\leftarrow}$ if and only if $\gamma_2(T^{n+2(3^{l-1})}y_0) = \boxed{\leftarrow}$. Using this and Remark 3.37, we obtain that

$$S^{DM}_{\{0\}}(x,m) = +S^{DM}_{2^{l-1}+1}(x,m) + 2(3^{l-1}).$$

Therefore,

$$\overline{D}(S^{DM}_{\{0\}}(x,m)) = \overline{D}(S^{DM}_{\{2^{l-1}+1\}}(x,m)).$$

Notice that for any $y \in B_m(x)$ and $i \geq 0$ we have that $T^{n+1}y_{-(i+1)} = \boxed{\leftarrow}$ if and only if $T^n y_{-i} = \boxed{\leftarrow}$. This implies that

$$S^{DM}_{\{-i\}}(x,m) = S^{DM}_{\{0\}}(x,m) + i.$$

Hence,

$$\overline{D}(S^{DM}_{\{-i\}}(x,m)) = \overline{D}(S^{DM}_{\{2^{l-1}+1\}}(x,m)).$$

Lastly, let $1 \leq i \leq 2^{l-1}$. By Lemma 3.29, we have that

- $\gamma_1(T^n x_1) = \boxed{0}$ for all $0 \leq n < 3^{l-1}$;
- $\gamma_1(T^n x_1) = \boxed{1}$ for all $3^{l-1} \leq n < 2(3^{l-1})$;
- $\gamma_1(T^n x_1) = \boxed{2}$ for all $2(3^{l-1}) \leq n < 3^l$.

This implies that for any $y \in B_m(x)$ we have that $\gamma_2(T^n y_i) = \boxed{\leftarrow}$ for at most $2(3^{l-1})$ consecutive n. Hence, by Remark 3.37 we can conclude that

$$\sharp(S^{DM}_{\{i\}}(x,m) \cap [0,(k+1)3^{l+1}]) \leq (2(3^j)+1)\sharp(S^{DM}_{\{2^{l-1}+1\}}(x,m) \cap [0,(k+1)3^{l+1}]),$$

for all $k \geq 0$. Therefore,

$$\overline{D}(S^{DM}_{\{i\}}(x,m)) \leq (2(3^{l-1})+1)\overline{D}(S^{DM}_{\{2^{l-1}+1\}}(x,m)).$$

$\square$

One of the objectives in the construction of this example is to show that diam-mean equicontinuous points exist and that they are dense. Hence the importance of the following proposition.

Proposition 3.39. *Let $k > 0$, $w \in A^k$ and*

$$x := {}^\infty\square.w\square\ \boxed{0}\ \square\ \boxed{0}^2\ \square\ \boxed{0}^{2^2} \cdots \boxed{0}^{2^n} \cdots.$$

We have that x is a diam-mean equicontinuity point.

Proof. We will prove that x is a diam-mean equicontinuity point with the use of Proposition 2.22. Let $m \geq 0$. First notice that, without loss of generality we may assume that $\gamma_2(w_i) = \square$ for every $1 \leq i \leq k$ (from Lemma 3.34 there exists $M > 0$ such that $\gamma_2(T^M x_i) = \square$ for all $0 \leq i < k$). Let $l > 0$ so that $k < 2^l$ and

$$\frac{2(3^{l-1}+1)}{3^{l+1}} \leq \frac{1}{2^{m+2}}.$$

Let $k \leq j \leq k + l + \sum_{i=0}^{l-1} 2^i$. By applying Lemma 3.38 recurrently (and using that $k < 2^l$), we have that

$$\overline{D}(S^{DM}_{\{-j,j\}}(x, k+l+\textstyle\sum_{i=0}^{l} 2^i)) \;\; \leq \;\; 2(3^{l-1}+1)\overline{D}(S^{DM}_{\{|w|+l+\sum_{i=0}^{l-1} 2^i\}}(x, k+l+\textstyle\sum_{i=0}^{l} 2^i)).$$

Using that $k < 2^l$ and similar techniques as the ones used in the proof of Lemma 3.38 we can conclude the same inequality for $j \leq k + l + \sum_{i=0}^{l-1} 2^i$. By Lemma 3.36 and the choice of l, we obtain that

$$\begin{aligned}\overline{D}(S^{DM}_{\{-j,j\}}(x, k+l+\textstyle\sum_{i=0}^{l} 2^i)) \;\; &\leq \;\; \tfrac{2(3^{l-1}+1)}{3^{l+1}} \\ &\leq \;\; \tfrac{1}{2^{m+2}}.\end{aligned}$$

Therefore, by Proposition 2.22, we have that x is a diam-mean equicontinuity point.

$\square$

We conclude this section with the following statement and the main purpose for building this example.

Theorem 3.40. *$(A^{\mathbb{Z}}, T)$ is almost diam-mean equicontinuous but not almost equicontinuous (and also not mean equicontinuous).*

Proof. From Proposition 3.39 we have EQ^{DM} is dense. Hence, by Proposition 1.12, EQ^{DM} is residual. So, $(A^{\mathbb{Z}}, T)$ is almost diam-mean equicontinuous. By Proposition 3.35 there are no equicontinuity points. Therefore, $(A^{\mathbb{Z}}, T)$ is not almost equicontinuous. $\square$

The next result is not in [5].

Proposition 3.41. *$(A^{\mathbb{Z}}, T)$ is not cofinitely sensitive but it is syndetically sensitive.*

Proof. Let $w \in A^+$ such that $w = \square\,\boxed{0}$. From Lemma 3.36 we have that $N_T([w], 1)$ is not a cofinite set. Let $w \in A^+$. Without of lost of generality, let us assume that $w_i \in \{\square, \boxed{0}, \boxed{1}, \boxed{2}\}$. Let us fix $x = {}^{\infty}\boxed{0}.w\boxed{0}^{\infty}$ and ${}^{\infty}\square.w\boxed{\leftarrow}^{\infty}$. By Lemma 3.38 we have that $\overline{D}(S^{DM}_{\{-i,i\}}(x, |w|)) > 0$ for all $0 \leq i < |w|$. Hence, $N_T([w]_0, 1)$ is syndetic. Therefore $(A^{\mathbb{Z}}, T)$ is syndetically sensitive. $\square$

3.5 Diam-mean sensitivity

All the results in this section and their proofs, except for Corollary 3.44 and Proposition 3.45, are contained in [5].

In Section 3.2, we construct Pacman Level 2 to show that Kurka's dichotomy does not hold for the mean version. Similarly, we will construct a CA that shows that the diam-mean version of Kurka's dichotomy does not hold.

Let $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ the CA from the Section 3.3 and $A_3 = \{a, b, c\}$. We define $T_3 : A_3^{\mathbb{Z}} \to A_3^{\mathbb{Z}}$ as

$$T_3 x_i \quad = \quad \begin{cases} a & if \quad x_i = a; \\ b & if \quad x_i = c; \\ c & if \quad x_i = b. \end{cases}$$

We set $A_S := A \times A_3$. For every $x \in A_S$, we have that x^1 is the component on A and x^2 is the component on A_3. Let $id : A_3^{\mathbb{Z}} \to A_3^{\mathbb{Z}}$ be the identity function and $T_S : A_S^{\mathbb{Z}} \to A_S^{\mathbb{Z}}$ a CA defined locally with

$$T_S x_i \quad = \quad \begin{cases} (Tx_i^1, id(x_i^2)) & \text{if } \gamma_2(x_i^1) = \boxed{\leftarrow}; \\ (Tx_i^1, T_3 x_i^2) & \text{otherwise.} \end{cases}$$

Thus, on A, T_S behaves exactly as T, and on A_3, as T_3 except if there is an arrow on the first coordinate. When this happens the periodicity on b and c changes.

In the next lemma, we establish that the set of diam-mean equicontinuity points is not dense.

Lemma 3.42. *Let $m > 0$ and $w \in A_S^m$ such that $w_0 = (\square, b)$. Then, there exist $x, y \in A_S^{\mathbb{Z}}$ such that*

$$x_{[0,|w|-1]} = y_{[0,|w|-1]} = w$$

and the set

$$\mathbb{Z}_{n\geq 0} \setminus \{n \in \mathbb{Z}_{n\geq 0} : T_S^n x_0 \neq T_S^n y_0\}$$

is finite.

Proof. Let $w \in A_S^m$ as in the hypothesis of the statement. Let us define

$$x = {}^{\infty}(\square, a).w(\boxed{\leftarrow}, a)(\square, a)^{\infty}$$

and

$$y = {}^{\infty}(\square, a).w(\square, a)^{\infty}.$$

Using Lemma 3.34 we can assume, without loss of generality, that

$$w_i \in \{(p, q) : p \in \{\square, \boxed{0}, \boxed{1}, \boxed{2}\} \wedge q \in A_3\}.$$

Now, there exists $N > 0$ such that $T_S^N x_0 = (\boxed{\leftarrow}, q)$, where $q \in \{b, c\}$. We have two cases to prove.

Case 1: $T_S^N x_0 = (\boxed{\leftarrow}, b)$.
This implies that $T_S^{N+1} x_0 = (\square, b)$. Meanwhile, $T_S^{N+1} y_0 = (\square, c)$. Therefore, we can easily see that $T_S^{N+i} x_0 \neq T_S^{N+i} y_0$, for all $i > 0$.

Case 2: $T_S^N x_0 = (\boxed{\leftarrow}, c)$.
Again we have that $T_S^{N+1} x_0 = (\square, c)$, so $T_S^{N+i} x_0 \neq T_S^{N+i} y_0$ for all $i \geq 0$.

$\square$

Notice that for all $\varepsilon > 0$, any $y \in B_\varepsilon(x)$, where $x_0 = (\square, b)$, is not a diam-mean equicontinuity point.

Theorem 3.43. *$(A_S^{\mathbb{Z}}, T_S)$ is neither diam-mean sensitive nor almost diam-mean equicontinuous.*

Proof. From Lemma 3.42 we conclude that for every $x \in A_S^{\mathbb{Z}}$ such that $x_0 = (\square, b)$, we have that x is not a diam-mean equicontinuity point.

From Proposition 3.39, we have that

$$x := {}^{\infty}(\square, a).(\square, a)(\boxed{0}, a)(\square, a)(\boxed{0}, a)^2(\square, a)(\boxed{0}, a)^{2^2} \cdots (\square, a)(\boxed{0}, a)^{2^l} \cdots$$

is a diam-mean equicotinuity point. Hence, for all $\varepsilon > 0$ there exists $l \geq 0$ such that

$$\limsup_{n\to\infty} \frac{\sum_{j=0}^{n} \operatorname{diam}(T_S^j(B_{l+\sum_{i=0}^{l} 2^i}(x)))}{n+1} < \varepsilon.$$

Therefore, $(A_S^{\mathbb{Z}}, T_S)$ is not diam-mean sensitive. $\square$

From Pacman CA Level 2 we have that there are CA that are neither almost mean equicontinuous nor mean sensitive. Now, from $(A_S^{\mathbb{Z}}, T_S)$, we have that there exist CA such that they are neither almost diam-mean equicontinuous nor diam-mean sensitive. Is $(A_S^{\mathbb{Z}}, T_S)$ mean sensitive or almost mean equicontinuous? The answer is none, as we will see in the next corollary.

Corollary 3.44. *$(A_S^{\mathbb{Z}}, T_S)$ is neither mean sensitive nor almost mean equicontinuous.*

Proof. From Lemma 3.42 we have that $(A_S^{\mathbb{Z}}, T_S)$ is not almost mean equicontinuous; and by the second part of the proof of Theorem 3.18, we have that $(A_S^{\mathbb{Z}}, T_S)$ is not mean sensitive. $\square$

As we saw, $(A_S^{\mathbb{Z}}, T_S)$ is neither mean sensitive nor almost mean equicontinuous. So, where does this AC lie?

Proposition 3.45. *$(A_S^{\mathbb{Z}}, T_S)$ is not confinitely sensitive. However, $(A_S^{\mathbb{Z}}, T_S)$ is syndetically sensitive.*

Proof. Let $w \in A_S^+$ such that $w = (\square, a)(\boxed{0}, a)(\square, a)$. From Lemma 3.36, we have that any $x \in [w]$ satisfies that

$$\overline{D}(S_{\{0\}}^{DM}(x, |w|)) \leq \frac{1}{3}.$$

Hence, we have that $(A_S^{\mathbb{Z}}, T_S)$ is not confinitely sensitive.

Let $w \in A_S^+$. Without loss of generality, let us assume that $w_i \in \{(p, q) : p \in \{\square, \boxed{0}, \boxed{1}, \boxed{2}\} \wedge q \in A_3\}$ for all $0 \leq i < |w|$. Let fix $x = {}^{\infty}(\square, a).w(\square, a)^{\infty}$ and $y = {}^{\infty}(\square, a).w(\boxed{\leftarrow}, a)^{\infty}$. By the above, we have that $\overline{D}(S_{\{i\}}^{DM}(x, |w|)) > 0$ for all $0 \leq i < |w|$. Hence, $N_{T_S}([w], 1)$ is syndetic. Therefore, $(A_S^{\mathbb{Z}}, T_S)$ is syndetically sensitive. $\square$

Chapter 4

Final comments

For this final chapter we will see some questions derived from the results of our research and other questions related to the topic of our research. We will divide the chapter into small sections to give context to the questions raised.

4.1 Minimal TDS

A minimal TDS is mean equicontinuous if and only if it is not mean sensitive (see [22, 13]. Considering Proposition 2.10, we ask.

Question 4.1. *Does there exist a minimal subshift* (X, σ) *and a CA* (X, T) *that is neither mean equicontinuous nor mean sensitive?*

We remind the reader that a TDS is expansive if ther exists $\varepsilon > 0$ such that for every $x, y \in X$ there exists $n > 0$ such that $d(T^n x, T^n y) > \varepsilon$. We know that every expansive CA is sensitive, thus we ask.

Question 4.2. *Is every expansive CA mean sensitive?*

4.2 One dimensional CA

In Chapter 2 we saw some examples of Elementary Cellular Automata (ECA). We conjecture that a stronger dichotomy holds in this case.

Question 4.3. *Is every elementary CA that is not almost equicontinuous cofinitely sensitive?*

So far, we have seen that all the examples presented in this thesis are either cofinitely sensitive or syndetically sensitive. Even $(A_S^{\mathbb{Z}}, T_S)$ is neither almost mean equicontinuous nor diam-mean sensitive, but it is syndetically sensitive.

Question 4.4. *Does there exist a one-dimensional CA that is neither almost mean equicontinuous nor syndetically sensitive?*

Another more general question is the following:

Question 4.5. *Does there exist a CA that is almost mean equicontinuous that is niether almost diam-mean equicontinuous nor diam-mean sensitive?*

4.3 Monotone and conservative CA

Let $(A^{\mathbb{Z}}, T)$ be a CA, with $A = \{0, \dots, q-1\}$, with $q \in \mathbb{N}$, and let C_P be the set of all periodic configurations in $A^{\mathbb{Z}}$; for each $c \in C_P$ choose a period $p(c)$. Let ϕ be a function $\phi : A^b \to \mathbb{R}$, where b is a non-negative integer. Then ϕ is said to be a **non-increasing additive quantity** under T if and only if

$$\sum_{k=0}^{p(c)-1} \phi(T(c)_k, \dots, T(c)_{k+b-1}) \leq \sum_{k=0}^{p(c)-1} \phi(c_k, \dots, c_{k+b-1}),\ \forall c \in C_P. \tag{4.1}$$

Similarly, ϕ is said to be **non-decreasing additive quantity** if condition (4.1) holds with the inequality in the other direction. It is easy to see that ϕ is non-increasing if and only if -ϕ is non-decreasing. If ϕ is both non-increasing and non-decreasing, it is said to be a **conserved additive quantity**. We said that ϕ is **monotone** if it is either non-increasing or nondecreasing.

The Pacman CA is monotone but not conservative.

A CA is said to be conservative if A is a subset of the integers and the identity is conserved.

Question 4.6. *For conservative CA, is almost equicontinuity and not mean sensitivity equivalent?*

4.4 Measure sensitivity and equicontinuity

Definition 4.7. Let (X, T) be a TDS and μ a Borel probability measure on X.

1. We say (X, T) is **μ-equicontinuous** if for every $\tau > 0$ there exists a compact set $M \subset X$ such that $\mu(M) > 1 - \tau$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in M$ and $d(x, y) \leq \delta$ then $d(T^n x, T^n y) \leq \varepsilon$ $\forall n \in \mathbb{N}$.

2. We say (X, T) is **μ-sensitive** if there exists $\varepsilon > 0$ such that

$$\mu \times \mu(\{(x, y) \in X^2 : \exists n \in \mathbb{N} \text{ s.t. } d(T^n x, T^n y)\}) = 1.$$

Transitivity plays an important role when talking about the Akin-Auslander-Berg dichotomy (see Theorem 1.7). Now to talk about equicontinuity involving measures it is natural to introduce the concept of ergodicity which plays an important role for the Huang-Lu-Ye dichotomy.

Definition 4.8. Let (X, T) be a TDS and μ a Borel probability measure. We say that μ is **ergodic** if $\mu(A) = 0$ or 1 for every Borel set $A \subseteq X$ such that $T^{-1}(A) = A$.

Theorem 4.9. *[19] Let (X, T) be a TDS and μ a T-ergodic Borel probability measure on X. Then (X, T) is μ-equicontinuous if and only if it is not μ-sensitive.*

Just as we have Kurka's dichotomy for CA, we also have a dichotomy for CA with respect to μ-sensitive and μ-equicontinuous without assuming ergodicity with respect to the CA.

Theorem 4.10. *[16] Let $(A^{\mathbb{Z}}, T)$ be a CA and μ a σ-ergodic Borel probability measure on X. Then (X, T) is μ-equicontinuous if and only if it is not μ-sensitive.*

Note that for the theorem above μ may not be T-ergodic.

Naturally we may define the mean versions of μ-sensitive and μ-equicontinuous.

Definition 4.11. Let (X,T) be a TDS and μ a Borel probability measure on X.

1. We say (X,T) is **μ-mean equicontinuous** if for every $\tau > 0$ there exists a compact set $M \subset X$ such that $\mu(M) > 1-\tau$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in M$ and $d(x,y) \leq \delta$ then
$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) \leq \varepsilon.$$

2. We say (X,T) is **μ-mean sensitive** if there exists $\varepsilon > 0$ such that
$$\mu \times \mu(\left\{(x,y) \in X^2 : \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(T^i x, T^i y) > \varepsilon\right\}) = 1.$$

We now have the analog of Theorem 4 for the mean μ version (the García-Ramos dichotomy).

Theorem 4.12. *[13] Let (X,T) be a TDS and μ a T-ergodic Borel probability measure on X. Then (X,T) is μ-mean equicontinuous if and only if it is not μ-mean sensitive.*

Definition 4.13. Let A be an finite alphabet and $m > 0$. The **Bernoulli measures** are defined on cylinder sets and then extended to the whole sigma-algebra; that is, let $w \in A^n$, where $n > 0$, $[w]_m := \{x \in A^{\mathbb{Z}} : x_{[m,m+|w|)} = w\}$ we have that
$$\mu([w]_m) = \prod_{i=1}^{|w|} p_i,$$
where p_i is the probability of w_i for $1 \leq i \leq |w|$.

Having these tools and influenced by the work in these thesis we have the following questions.

Question 4.14. *Is there a one-dimensional CA and a Bernoulli probability measure μ, such that the CA is μ-mean equicontinuous but not μ-equicontinuous?*

Question 4.15. *Is there a one-dimensional CA and a Bernoulli probability measure μ, such that the CA is neither μ-mean sensitive nor μ-mean equicontinuous?*

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